11

Parametric Equations and Polar Coordinates

OVERVIEW In this chapter we study new ways to define curves in the plane. Instead of thinking of a curve as the graph of a function or equation, we consider a more general way of thinking of a curve as the path of a moving particle whose position is changing over time. Then each of the x- and y-coordinates of the particle's position becomes a function of a third variable t. We can also change the way in which points in the plane themselves are described by using *polar coordinates* rather than the rectangular or Cartesian system. Both of these new tools are useful for describing motion, like that of planets and satellites, or projectiles moving in the plane or space. In addition, we review the geometric definitions and standard equations of parabolas, ellipses, and hyperbolas. These curves are called *conic sections*, or *conics*, and model the paths traveled by projectiles, planets, or any other object moving under the sole influence of a gravitational or electromagnetic force.

11.1 Parametrizations of Plane Curves

In previous chapters, we have studied curves as the graphs of functions or of equations involving the two variables x and y. We are now going to introduce another way to describe a curve by expressing both coordinates as functions of a third variable t.

Parametric Equations

Figure 11.1 shows the path of a moving particle in the xy-plane. Notice that the path fails the vertical line test, so it cannot be described as the graph of a function of the variable x. However, we can sometimes describe the path by a pair of equations, $x = f(t)$ and $y = g(t)$, where f and g are continuous functions. When studying motion, t usually denotes time. Equations like these describe more general curves than those described by a single function, and they provide not only the graph of the path traced out but also the location of the particle $(x, y) = (f(t), g(t))$ at any time t.

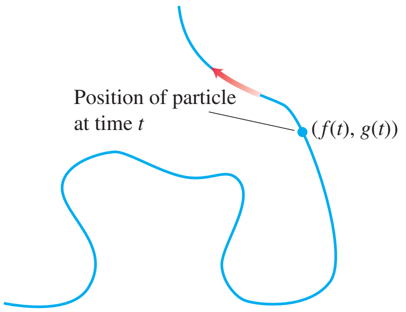


FIGURE 11.1 The curve or path traced by a particle moving in the xy-plane is not always the graph of a function or single equation.

DEFINITION If x and y are given as functions

$$x = f(t), \qquad y = g(t)$$

over an interval I of t-values, then the set of points $(x, y) = (f(t), g(t))$ defined by these equations is a **parametric curve**. The equations are **parametric equations** for the curve.

The variable t is a **parameter** for the curve, and its domain I is the **parameter interval**. If I is a closed interval, $a \le t \le b$, the point $(f(a), g(a))$ is the **initial point** of the curve and $(f(b), g(b))$ is the **terminal point**. When we give parametric equations and a parameter interval for a curve, we say that we have **parametrized** the curve. The equations and interval together constitute a **parametrization** of the curve. A given curve can be represented by different sets of parametric equations. (See Exercises 19 and 20.)

EXAMPLE 1 Sketch the curve defined by the parametric equations

$$x = t^2, \qquad y = t + 1, \qquad -\infty < t < \infty.$$

Solution We make a brief table of values (Table 11.1), plot the points (x, y), and draw a smooth curve through them (Figure 11.2). Each value of t gives a point (x, y) on the curve, such as $t = 1$ giving the point (1, 2) recorded in Table 11.1. If we think of the curve as the path of a moving particle, then the particle moves along the curve in the direction of the arrows shown in Figure 11.2. Although the time intervals in the table are equal, the consecutive points plotted along the curve are not at equal arc length distances. The reason for this is that the particle slows down as it gets nearer to the y-axis along the lower branch of the curve as t increases, and then speeds up after reaching the y-axis at (0, 1) and moving along the upper branch. Since the interval of values for t is all real numbers, there is no initial point and no terminal point for the curve. ■

TABLE 11.1 Values of $x = t^2$ and $y = t + 1$ for selected values of t.

t	x	y
−3	9	−2
−2	4	−1
−1	1	0
0	0	1
1	1	2
2	4	3
3	9	4

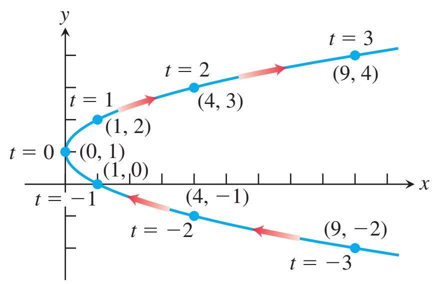


FIGURE 11.2 The curve given by the parametric equations $x = t^2$ and $y = t + 1$ (Example 1).

EXAMPLE 2 Identify geometrically the curve in Example 1 (Figure 11.2) by eliminating the parameter t and obtaining an algebraic equation in x and y.

Solution We solve the equation $y = t + 1$ for the parameter t and substitute the result into the parametric equation for x. This procedure gives $t = y - 1$ and

$$x = t^2 = (y - 1)^2 = y^2 - 2y + 1.$$

The equation $x = y^2 - 2y + 1$ represents a parabola, as displayed in Figure 11.2. It is sometimes quite difficult, or even impossible, to eliminate the parameter from a pair of parametric equations, as we did here. ■

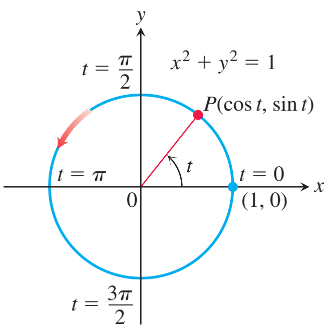


Figure 11.3 The equations $x = \cos t$ and $y = \sin t$ describe motion on the circle $x^2 + y^2 = 1$. The arrow shows the direction of increasing t (Example 3).

EXAMPLE 3 Graph the parametric curves

(a) $x = \cos t, \qquad y = \sin t, \qquad 0 \le t \le 2\pi.$

(b) $x = a\cos t, \qquad y = a\sin t, \qquad 0 \le t \le 2\pi.$

Solution

(a) Since $x^2 + y^2 = \cos^2 t + \sin^2 t = 1$, the parametric curve lies along the unit circle $x^2 + y^2 = 1$. As t increases from 0 to 2π, the point $(x, y) = (\cos t, \sin t)$ starts at (1, 0) and traces the entire circle once counterclockwise (Figure 11.3).

(b) For $x = a\cos t$, $y = a\sin t$, $0 \le t \le 2\pi$, we have $x^2 + y^2 = a^2\cos^2 t + a^2\sin^2 t = a^2$. The parametrization describes a motion that begins at the point $(a, 0)$ and traverses the circle $x^2 + y^2 = a^2$ once counterclockwise, returning to $(a, 0)$ at $t = 2\pi$. The graph is a circle centered at the origin with radius $r = |a|$ and coordinate points $(a\cos t, a\sin t)$. ■

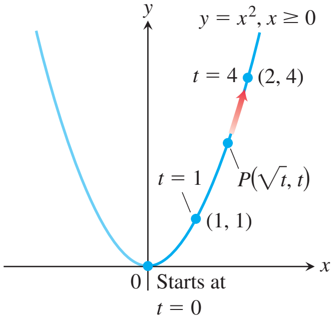


FIGURE 11.4 The equations $x = \sqrt{t}$ and $y = t$ and the interval $t \ge 0$ describe the path of a particle that traces the right-hand half of the parabola $y = x^2$ (Example 4).

EXAMPLE 4 The position $P(x, y)$ of a particle moving in the xy-plane is given by the equations and parameter interval

$$x = \sqrt{t}, \qquad y = t, \qquad t \ge 0.$$

Identify the path traced by the particle and describe the motion.

Solution We try to identify the path by eliminating t between the equations $x = \sqrt{t}$ and $y = t$, which might produce a recognizable algebraic relation between x and y. We find that

$$y = t = \left(\sqrt{t}\right)^2 = x^2.$$

Thus, the particle's position coordinates satisfy the equation $y = x^2$, so the particle moves along the parabola $y = x^2$.

It would be a mistake, however, to conclude that the particle's path is the entire parabola $y = x^2$; it is only half the parabola. The particle's x-coordinate is never negative. The particle starts at (0, 0) when $t = 0$ and rises into the first quadrant as t increases (Figure 11.4). The parameter interval is $[0, \infty)$ and there is no terminal point. ■

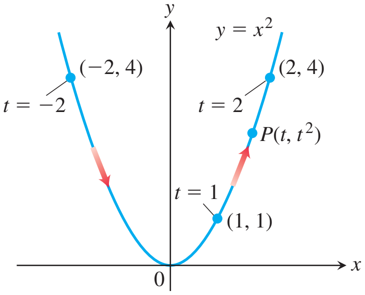


FIGURE 11.5 The path defined by $x = t, y = t^2, -\infty < t < \infty$ is the entire parabola $y = x^2$ (Example 5).

The graph of any function $y = f(x)$ can always be given a **natural parametrization** $x = t$ and $y = f(t)$. The domain of the parameter in this case is the same as the domain of the function f.

EXAMPLE 5 A parametrization of the graph of the function $f(x) = x^2$ is given by

$$x = t, \qquad y = f(t) = t^2, \qquad -\infty < t < \infty.$$

When $t \ge 0$, this parametrization gives the same path in the xy-plane as we had in Example 4. However, since the parameter t here can now also be negative, we obtain the left-hand part of the parabola as well; that is, we have the entire parabolic curve. For this parametrization, there is no starting point and no terminal point (Figure 11.5). ■

Notice that a parametrization also specifies *when* (the value of the parameter) a particle moving along the curve is *located* at a specific point along the curve. In Example 4, the point (2, 4) is reached when $t = 4$; in Example 5, it is reached "earlier" when $t = 2$. You can see the implications of this aspect of parametrizations when considering the possibility of two objects coming into collision: they have to be at the exact same location point $P(x, y)$ for some (possibly different) values of their respective parameters. We will say more about this aspect of parametrizations when we study motion in Chapter 13.

EXAMPLE 6 Find a parametrization for the line through the point (a, b) having slope m.

Solution A Cartesian equation of the line is $y - b = m(x - a)$. If we set the parameter $t = x - a$, we find that $x = a + t$ and $y - b = mt$. That is,

$$x = a + t, \qquad y = b + mt, \qquad -\infty < t < \infty$$

parametrizes the line. This parametrization differs from the one we would obtain by the natural parametrization in Example 5 when $t = x$. However, both parametrizations describe the same line. ■

TABLE 11.2 Values of $x = t + (1/t)$ and $y = t - (1/t)$ for selected values of t.

t	$1/t$	x	y
0.1	10.0	10.1	−9.9
0.2	5.0	5.2	−4.8
0.4	2.5	2.9	−2.1
1.0	1.0	2.0	0.0
2.0	0.5	2.5	1.5
5.0	0.2	5.2	4.8
10.0	0.1	10.1	9.9

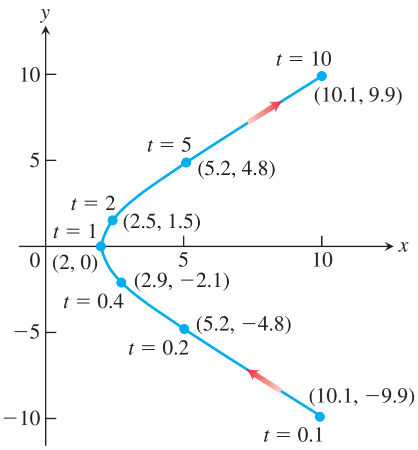


FIGURE 11.6 The curve for $x = t + (1/t)$, $y = t - (1/t)$, $t > 0$ in Example 7. (The part shown is for $0.1 \le t \le 10$.)

EXAMPLE 7 Sketch and identify the path traced by the point $P(x, y)$ if

$$x = t + \frac{1}{t}, \qquad y = t - \frac{1}{t}, \qquad t > 0.$$

Solution We make a brief table of values in Table 11.2, plot the points, and draw a smooth curve through them, as we did in Example 1. Next we eliminate the parameter t from the equations. The procedure is more complicated than in Example 2. Taking the difference between x and y as given by the parametric equations, we find that

$$x - y = \left(t + \frac{1}{t}\right) - \left(t - \frac{1}{t}\right) = \frac{2}{t}.$$

If we add the two parametric equations, we get

$$x + y = \left(t + \frac{1}{t}\right) + \left(t - \frac{1}{t}\right) = 2t.$$

We can then eliminate the parameter t by multiplying these last equations together:

$$(x - y)(x + y) = \left(\frac{2}{t}\right)(2t) = 4,$$

or, expanding the expression on the left-hand side, we obtain a standard equation for a hyperbola (reviewed in Section 11.6):

$$x^2 - y^2 = 4. \qquad (1)$$

Thus the coordinates of all the points $P(x, y)$ described by the parametric equations satisfy Equation (1). However, Equation (1) does not require that the x-coordinate be positive. So there are points (x, y) on the hyperbola that do not satisfy the parametric equation $x = t + (1/t)$, $t > 0$, for which x is always positive. That is, the parametric equations do not yield any points on the left branch of the hyperbola given by Equation (1), points where the x-coordinate would be negative. For small positive values of t, the path lies in the fourth quadrant and rises into the first quadrant as t increases, crossing the x-axis when $t = 1$ (see Figure 11.6). The parameter domain is $(0, \infty)$ and there is no starting point and no terminal point for the path. ■

Examples 4, 5, and 6 illustrate that a given curve, or portion of it, can be represented by different parametrizations. In the case of Example 7, we can also represent the right-hand branch of the hyperbola by the parametrization

$$x = \sqrt{4 + t^2}, \qquad y = t, \qquad -\infty < t < \infty,$$

which is obtained by solving Equation (1) for $x \ge 0$ and letting y be the parameter. Still another parametrization for the right-hand branch of the hyperbola given by Equation (1) is

$$x = 2 \sec t, \qquad y = 2 \tan t, \qquad -\frac{\pi}{2} < t < \frac{\pi}{2}.$$

This parametrization follows from the trigonometric identity $\sec^2 t - \tan^2 t = 1$, so

$$x^2 - y^2 = 4 \sec^2 t - 4 \tan^2 t = 4(\sec^2 t - \tan^2 t) = 4.$$

As t runs between $-\pi/2$ and $\pi/2$, $x = \sec t$ remains positive and $y = \tan t$ runs between $-\infty$ and ∞, so P traverses the hyperbola's right-hand branch. It comes in along the branch's lower half as $t \to 0^-$, reaches $(2, 0)$ at $t = 0$, and moves out into the first quadrant as t increases steadily toward $\pi/2$. This is the same hyperbola branch for which a portion is shown in Figure 11.6.

HISTORICAL BIOGRAPHY
Christian Huygens
(1629–1695)

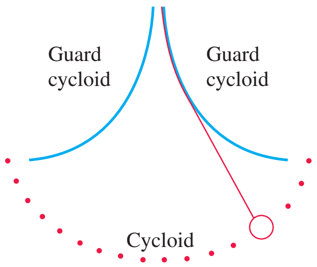


FIGURE 11.7 In Huygens' pendulum clock, the bob swings in a cycloid, so the frequency is independent of the amplitude.

Cycloids

The problem with a pendulum clock whose bob swings in a circular arc is that the frequency of the swing depends on the amplitude of the swing. The wider the swing, the longer it takes the bob to return to center (its lowest position).

This does not happen if the bob can be made to swing in a *cycloid*. In 1673, Christian Huygens designed a pendulum clock whose bob would swing in a cycloid, a curve we define in Example 8. He hung the bob from a fine wire constrained by guards that caused it to draw up as it swung away from center (Figure 11.7), and we describe the path parametrically in the next example.

EXAMPLE 8 A wheel of radius a rolls along a horizontal straight line. Find parametric equations for the path traced by a point P on the wheel's circumference. The path is called a **cycloid**.

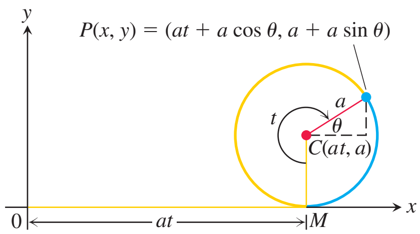


FIGURE 11.8 The position of $P(x, y)$ on the rolling wheel at angle t (Example 8).

Solution We take the line to be the x-axis, mark a point P on the wheel, start the wheel with P at the origin, and roll the wheel to the right. As parameter, we use the angle t through which the wheel turns, measured in radians. Figure 11.8 shows the wheel a short while later when its base lies at units from the origin. The wheel's center C lies at (at, a) and the coordinates of P are

$$x = at + a\cos\theta, \qquad y = a + a\sin\theta.$$

To express θ in terms of t, we observe that $t + \theta = 3\pi/2$ in the figure, so that

$$\theta = \frac{3\pi}{2} - t.$$

This makes

$$\cos\theta = \cos\left(\frac{3\pi}{2} - t\right) = -\sin t, \qquad \sin\theta = \sin\left(\frac{3\pi}{2} - t\right) = -\cos t.$$

The equations we seek are

$$x = at - a\sin t, \qquad y = a - a\cos t.$$

These are usually written with the a factored out:

$$x = a(t - \sin t), \qquad y = a(1 - \cos t). \tag{2}$$

Figure 11.9 shows the first arch of the cycloid and part of the next. ■

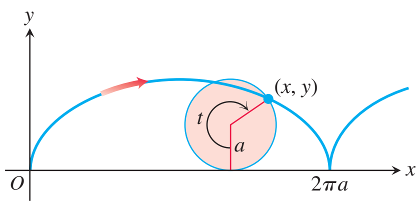


FIGURE 11.9 The cycloid curve $x = a(t - \sin t)$, $y = a(1 - \cos t)$, for $t \geq 0$.

Brachistochrones and Tautochrones

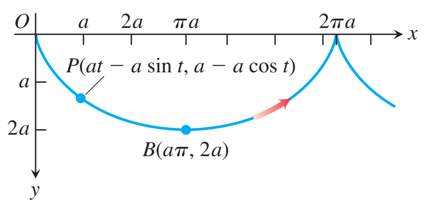


FIGURE 11.10 Turning Figure 11.9 upside down, the y-axis points downward, indicating the direction of the gravitational force. Equations (2) still describe the curve parametrically.

If we turn Figure 11.9 upside down, Equations (2) still apply and the resulting curve (Figure 11.10) has two interesting physical properties. The first relates to the origin O and the point B at the bottom of the first arch. Among all smooth curves joining these points, the cycloid is the curve along which a frictionless bead, subject only to the force of gravity, will slide from O to B the fastest. This makes the cycloid a **brachistochrone** ("brah-*kiss*-toe-krone"), or shortest-time curve for these points. The second property is that even if you start the bead partway down the curve toward B, it will still take the bead the same amount of time to reach B. This makes the cycloid a **tautochrone** ("*taw*-toe-krone"), or same-time curve for O and B.

Are there any other brachistochrones joining O and B, or is the cycloid the only one? We can formulate this as a mathematical question in the following way. At the start, the kinetic energy of the bead is zero, since its velocity (speed) is zero. The work done by gravity in moving the bead from (0, 0) to any other point (x, y) in the plane is mgy, and this must equal the change in kinetic energy. (See Exercise 23 in Section 6.5.) That is,

$$mgy = \frac{1}{2}mv^2 - \frac{1}{2}m(0)^2.$$

Thus, the speed of the bead when it reaches (x, y) has to be

$$v = \sqrt{2gy}.$$

That is,

$$\frac{ds}{dT} = \sqrt{2gy}$$

ds is the arc length differential along the bead's path and *T* represents time.

or

$$dT = \frac{ds}{\sqrt{2gy}} = \frac{\sqrt{1 + (dy/dx)^2}\,dx}{\sqrt{2gy}}. \tag{3}$$

The time T_f it takes the bead to slide along a particular path $y = f(x)$ from O to $B(a\pi, 2a)$ is

$$T_f = \int_{x=0}^{x=a\pi} \sqrt{\frac{1 + (dy/dx)^2}{2gy}}\,dx. \tag{4}$$

What curves $y = f(x)$, if any, minimize the value of this integral?

At first sight, we might guess that the straight line joining O and B would give the shortest time, but perhaps not. There might be some advantage in having the bead fall vertically at first to build up its speed faster. With a higher speed, the bead could travel a longer path and still reach B first. Indeed, this is the right idea. The solution, from a branch of mathematics known as the *calculus of variations*, is that the original cycloid from O to B is the one and only brachistochrone for O and B (Figure 11.11).

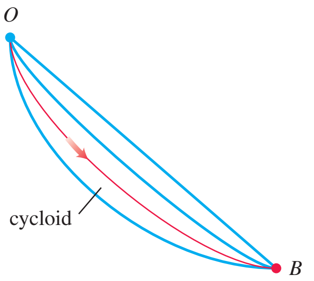


FIGURE 11.11 The cycloid is the unique curve which minimizes the time it takes for a frictionless bead to slide from point O to B.

In the next section we show how to find the arc length differential ds for a parametrized curve. Once we know how to find ds, we can calculate the time given by the right-hand side of Equation (4) for the cycloid. This calculation gives the amount of time it takes a frictionless bead to slide down the cycloid to B after it is released from rest at O. The time turns out to be equal to $\pi\sqrt{a/g}$, where a is the radius of the wheel defining the particular cycloid. Moreover, if we start the bead at some lower point on the cycloid, corresponding to a parameter value $t_0 > 0$, we can integrate the parametric form of $ds/\sqrt{2gy}$ in Equation (3) over the interval $[t_0, \pi]$ to find the time it takes the bead to reach the point B. That calculation results in the same time $T = \pi\sqrt{a/g}$. It takes the bead the same amount of time to reach B no matter where it starts, which makes the cycloid a tautochrone. Beads starting simultaneously from O, A, and C in Figure 11.12, for instance, will all reach B at exactly the same time. This is the reason why Huygens' pendulum clock in Figure 11.7 is independent of the amplitude of the swing.

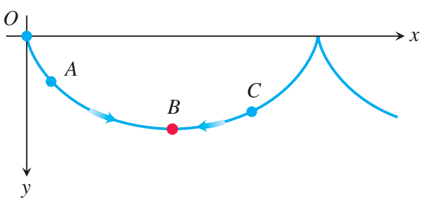


FIGURE 11.12 Beads released simultaneously on the upside-down cycloid at O, A, and C will reach B at the same time.

Exercises 11.1

Finding Cartesian from Parametric Equations

Exercises 1–18 give parametric equations and parameter intervals for the motion of a particle in the xy-plane. Identify the particle's path by finding a Cartesian equation for it. Graph the Cartesian equation. (The graphs will vary with the equation used.) Indicate the portion of the graph traced by the particle and the direction of motion.

1. $x = 3t, \quad y = 9t^2, \quad -\infty < t < \infty$
2. $x = -\sqrt{t}, \quad y = t, \quad t \geq 0$
3. $x = 2t - 5, \quad y = 4t - 7, \quad -\infty < t < \infty$
4. $x = 3 - 3t, \quad y = 2t, \quad 0 \leq t \leq 1$
5. $x = \cos 2t, \quad y = \sin 2t, \quad 0 \leq t \leq \pi$
6. $x = \cos(\pi - t), \quad y = \sin(\pi - t), \quad 0 \leq t \leq \pi$
7. $x = 4 \cos t, \quad y = 2 \sin t, \quad 0 \leq t \leq 2\pi$
8. $x = 4 \sin t, \quad y = 5 \cos t, \quad 0 \leq t \leq 2\pi$
9. $x = \sin t, \quad y = \cos 2t, \quad -\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$
10. $x = 1 + \sin t, \quad y = \cos t - 2, \quad 0 \leq t \leq \pi$
11. $x = t^2, \quad y = t^6 - 2t^4, \quad -\infty < t < \infty$
12. $x = \frac{t}{t - 1}, \quad y = \frac{t - 2}{t + 1}, \quad -1 < t < 1$
13. $x = t, \quad y = \sqrt{1 - t^2}, \quad -1 \leq t \leq 0$
14. $x = \sqrt{t + 1}, \quad y = \sqrt{t}, \quad t \geq 0$
15. $x = \sec^2 t - 1, \quad y = \tan t, \quad -\pi/2 < t < \pi/2$
16. $x = -\sec t, \quad y = \tan t, \quad -\pi/2 < t < \pi/2$
17. $x = -\cosh t, \quad y = \sinh t, \quad -\infty < t < \infty$
18. $x = 2 \sinh t, \quad y = 2 \cosh t, \quad -\infty < t < \infty$

Finding Parametric Equations

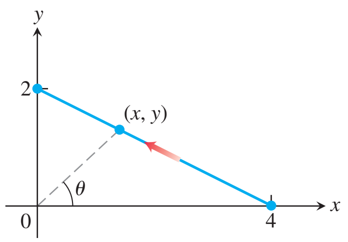
19. Find parametric equations and a parameter interval for the motion of a particle that starts at $(a, 0)$ and traces the circle $x^2 + y^2 = a^2$
    a. once clockwise.
    b. once counterclockwise.
    c. twice clockwise.
    d. twice counterclockwise.

    (There are many ways to do these, so your answers may not be the same as the ones in the back of the book.)
20. Find parametric equations and a parameter interval for the motion of a particle that starts at $(a, 0)$ and traces the ellipse $(x^2/a^2) + (y^2/b^2) = 1$
    a. once clockwise.
    b. once counterclockwise.
    c. twice clockwise.
    d. twice counterclockwise.

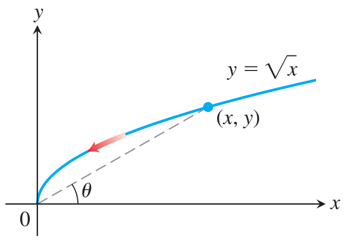
    (As in Exercise 19, there are many correct answers.)

In Exercises 21–26, find a parametrization for the curve.

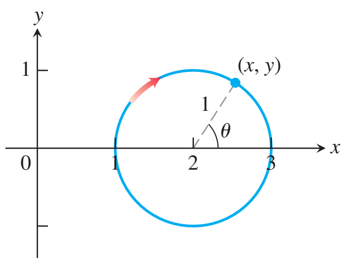
21. the line segment with endpoints $(-1, -3)$ and $(4, 1)$
22. the line segment with endpoints $(-1, 3)$ and $(3, -2)$
23. the lower half of the parabola $x - 1 = y^2$
24. the left half of the parabola $y = x^2 + 2x$
25. the ray (half line) with initial point $(2, 3)$ that passes through the point $(-1, -1)$
26. the ray (half line) with initial point $(-1, 2)$ that passes through the point $(0, 0)$
27. Find parametric equations and a parameter interval for the motion of a particle starting at the point $(2, 0)$ and tracing the top half of the circle $x^2 + y^2 = 4$ four times.
28. Find parametric equations and a parameter interval for the motion of a particle that moves along the graph of $y = x^2$ in the following way: Beginning at $(0, 0)$ it moves to $(3, 9)$, and then travels back and forth from $(3, 9)$ to $(-3, 9)$ infinitely many times.
29. Find parametric equations for the semicircle
$$x^2 + y^2 = a^2, \quad y > 0,$$
using as parameter the slope $t = dy/dx$ of the tangent to the curve at (x, y).
30. Find parametric equations for the circle
$$x^2 + y^2 = a^2,$$
using as parameter the arc length s measured counterclockwise from the point $(a, 0)$ to the point (x, y).
31. Find a parametrization for the line segment joining points $(0, 2)$ and $(4, 0)$ using the angle θ in the accompanying figure as the parameter.



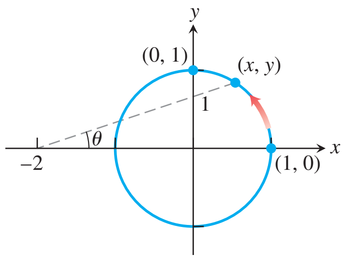
32. Find a parametrization for the curve $y = \sqrt{x}$ with terminal point $(0, 0)$ using the angle θ in the accompanying figure as the parameter.



33. Find a parametrization for the circle $(x - 2)^2 + y^2 = 1$ starting at $(1, 0)$ and moving clockwise once around the circle, using the central angle θ in the accompanying figure as the parameter.



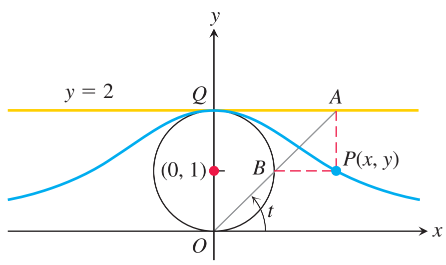
34. Find a parametrization for the circle $x^2 + y^2 = 1$ starting at $(1, 0)$ and moving counterclockwise to the terminal point $(0, 1)$, using the angle θ in the accompanying figure as the parameter.



35. The witch of Maria Agnesi The bell-shaped witch of Maria Agnesi can be constructed in the following way. Start with a circle of radius 1, centered at the point $(0, 1)$, as shown in the accompanying figure. Choose a point A on the line $y = 2$ and connect it to the origin with a line segment. Call the point where the segment crosses the circle B. Let P be the point where the vertical line through A crosses the horizontal line through B. The witch is the curve traced by P as A moves along the line $y = 2$. Find parametric equations and a parameter interval for the witch by expressing the coordinates of P in terms of t, the radian measure of the angle that segment O-A makes with the positive x-axis. The following equalities (which you may assume) will help.

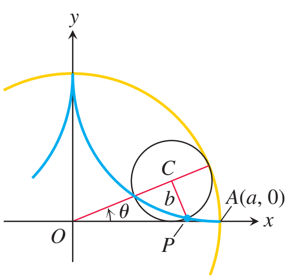
a. $x = AQ$ **b.** $y = 2 - AB \sin t$

c. $AB \cdot OA = (AQ)^2$

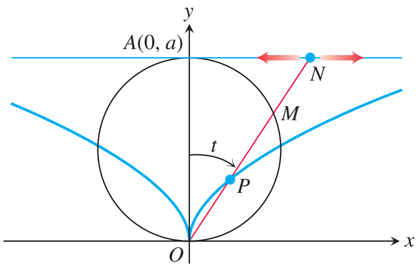


36. Hypocycloid When a circle rolls on the inside of a fixed circle, any point P on the circumference of the rolling circle describes a *hypocycloid*. Let the fixed circle be $x^2 + y^2 = a^2$, let the radius of the rolling circle be b, and let the initial position of the tracing point P be $A(a, 0)$. Find parametric equations for the hypocycloid, using as the parameter the angle θ from the positive x-axis to the line joining the circles' centers. In particular, if $b = a/4$, as in the accompanying figure, show that the hypocycloid is the astroid

$$x = a \cos^3 \theta, \quad y = a \sin^3 \theta.$$



37. As the point N moves along the line $y = a$ in the accompanying figure, P moves in such a way that $OP = MN$. Find parametric equations for the coordinates of P as functions of the angle t that the line ON makes with the positive y-axis.



38. Trochoids A wheel of radius a rolls along a horizontal straight line without slipping. Find parametric equations for the curve traced out by a point P on a spoke of the wheel b units from its center. As parameter, use the angle θ through which the wheel turns. The curve is called a *trochoid*, which is a cycloid when $b = a$.

Distance Using Parametric Equations

39. Find the point on the parabola $x = t, y = t^2, -\infty < t < \infty$, closest to the point $(2, 1/2)$. (*Hint:* Minimize the square of the distance as a function of t.)

40. Find the point on the ellipse $x = 2 \cos t, y = \sin t, 0 \le t \le 2\pi$ closest to the point $(3/4, 0)$. (*Hint:* Minimize the square of the distance as a function of t.)

T GRAPHER EXPLORATIONS

If you have a parametric equation grapher, graph the equations over the given intervals in Exercises 41–48.

41. Ellipse $x = 4 \cos t, \quad y = 2 \sin t,$ over

a. $0 \le t \le 2\pi$

b. $0 \le t \le \pi$

c. $-\pi/2 \le t \le \pi/2.$

42. Hyperbola branch $x = \sec t$ (enter as $1/\cos(t)$), $y = \tan t$ (enter as $\sin(t)/\cos(t)$), over

a. $-1.5 \le t \le 1.5$

b. $-0.5 \le t \le 0.5$

c. $-0.1 \le t \le 0.1.$

43. Parabola $x = 2t + 3, \quad y = t^2 - 1, \quad -2 \le t \le 2$

44. Cycloid $x = t - \sin t, \quad y = 1 - \cos t,$ over

a. $0 \le t \le 2\pi$

b. $0 \le t \le 4\pi$

c. $\pi \le t \le 3\pi.$

45. Deltoid

$$x = 2 \cos t + \cos 2t, \quad y = 2 \sin t - \sin 2t; \quad 0 \le t \le 2\pi$$

What happens if you replace 2 with −2 in the equations for x and y? Graph the new equations and find out.

46. A nice curve

$$x = 3 \cos t + \cos 3t, \quad y = 3 \sin t - \sin 3t; \quad 0 \le t \le 2\pi$$

What happens if you replace 3 with −3 in the equations for x and y? Graph the new equations and find out.

47. a. Epicycloid

$x = 9\cos t - \cos 9t, \quad y = 9\sin t - \sin 9t; \quad 0 \le t \le 2\pi$

b. Hypocycloid

$x = 8\cos t + 2\cos 4t, \quad y = 8\sin t - 2\sin 4t; \quad 0 \le t \le 2\pi$

c. Hypotrochoid

$x = \cos t + 5\cos 3t, \quad y = 6\cos t - 5\sin 3t; \quad 0 \le t \le 2\pi$

48. a. $x = 6\cos t + 5\cos 3t, \quad y = 6\sin t - 5\sin 3t;$ $0 \le t \le 2\pi$

b. $x = 6\cos 2t + 5\cos 6t, \quad y = 6\sin 2t - 5\sin 6t;$ $0 \le t \le \pi$

c. $x = 6\cos t + 5\cos 3t, \quad y = 6\sin 2t - 5\sin 3t;$ $0 \le t \le 2\pi$

d. $x = 6\cos 2t + 5\cos 6t, \quad y = 6\sin 4t - 5\sin 6t;$ $0 \le t \le \pi$

11.2 Calculus with Parametric Curves

In this section we apply calculus to parametric curves. Specifically, we find slopes, lengths, and areas associated with parametrized curves.

Tangents and Areas

A parametrized curve $x = f(t)$ and $y = g(t)$ is **differentiable** at t if f and g are differentiable at t. At a point on a differentiable parametrized curve where y is also a differentiable function of x, the derivatives dy/dt, dx/dt, and dy/dx are related by the Chain Rule:

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}.$$

If $dx/dt \neq 0$, we may divide both sides of this equation by dx/dt to solve for dy/dx.

Parametric Formula for dy/dx

If all three derivatives exist and $dx/dt \neq 0$,

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}. \tag{1}$$

If parametric equations define y as a twice-differentiable function of x, we can apply Equation (1) to the function $dy/dx = y'$ to calculate d^2y/dx^2 as a function of t:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(y') = \frac{dy'/dt}{dx/dt}. \qquad \text{Eq. (1) with } y' \text{ in place of } y$$

Parametric Formula for d^2y/dx^2

If the equations $x = f(t)$, $y = g(t)$ define y as a twice-differentiable function of x, then at any point where $dx/dt \neq 0$ and $y' = dy/dx$,

$$\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt}. \tag{2}$$

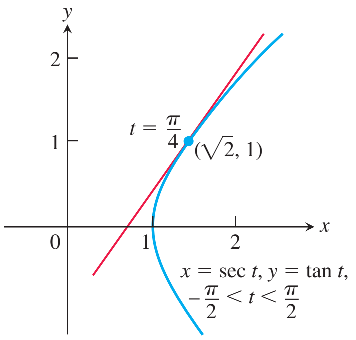


FIGURE 11.13 The curve in Example 1 is the right-hand branch of the hyperbola $x^2 - y^2 = 1$.

EXAMPLE 1 Find the tangent to the curve

$$x = \sec t, \qquad y = \tan t, \qquad -\frac{\pi}{2} < t < \frac{\pi}{2},$$

at the point $(\sqrt{2}, 1)$, where $t = \pi/4$ (Figure 11.13).

Solution The slope of the curve at t is

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\sec^2 t}{\sec t \tan t} = \frac{\sec t}{\tan t}. \qquad \text{Eq. (1)}$$

Setting t equal to $\pi/4$ gives

$$\left.\frac{dy}{dx}\right|_{t=\pi/4} = \frac{\sec(\pi/4)}{\tan(\pi/4)}$$

$$= \frac{\sqrt{2}}{1} = \sqrt{2}.$$

The tangent line is

$$y - 1 = \sqrt{2}\left(x - \sqrt{2}\right)$$
$$y = \sqrt{2}\,x - 2 + 1$$
$$y = \sqrt{2}\,x - 1.$$

EXAMPLE 2 Find d^2y/dx^2 as a function of t if $x = t - t^2$ and $y = t - t^3$.

Finding d^2y/dx^2 in Terms of t

1. Express $y' = dy/dx$ in terms of t.
2. Find dy'/dt.
3. Divide dy'/dt by dx/dt.

Solution

1. Express $y' = dy/dx$ in terms of t.

$$y' = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1 - 3t^2}{1 - 2t}$$

2. Differentiate y' with respect to t.

$$\frac{dy'}{dt} = \frac{d}{dt}\left(\frac{1 - 3t^2}{1 - 2t}\right) = \frac{2 - 6t + 6t^2}{(1 - 2t)^2} \qquad \text{Derivative Quotient Rule}$$

3. Divide dy'/dt by dx/dt.

$$\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt} = \frac{(2 - 6t + 6t^2)/(1 - 2t)^2}{1 - 2t} = \frac{2 - 6t + 6t^2}{(1 - 2t)^3} \qquad \text{Eq. (2)}$$

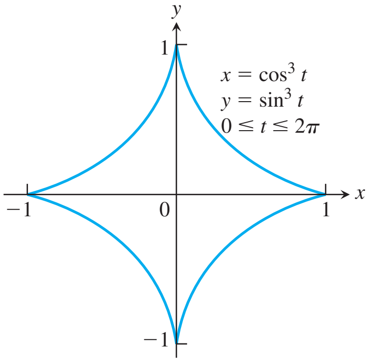


FIGURE 11.14 The astroid in Example 3.

EXAMPLE 3 Find the area enclosed by the astroid (Figure 11.14)

$$x = \cos^3 t, \qquad y = \sin^3 t, \qquad 0 \le t \le 2\pi.$$

Solution By symmetry, the enclosed area is 4 times the area beneath the curve in the first quadrant where $0 \le t \le \pi/2$. We can apply the definite integral formula for area studied in Chapter 5, using substitution to express the curve and differential dx in terms of the parameter t. So,

$$A = 4\int_0^1 y\,dx$$

$$= 4\int_0^{\pi/2} \sin^3 t \cdot 3\cos^2 t \sin t\,dt \qquad \text{Substitution for } y \text{ and } dx$$

$$= 12\int_0^{\pi/2} \left(\frac{1-\cos 2t}{2}\right)^2 \left(\frac{1+\cos 2t}{2}\right) dt \qquad \sin^4 t = \left(\frac{1-\cos 2t}{2}\right)^2$$

$$= \frac{3}{2}\int_0^{\pi/2} (1 - 2\cos 2t + \cos^2 2t)(1 + \cos 2t)\,dt \qquad \text{Expand squared term.}$$

$$= \frac{3}{2}\int_0^{\pi/2} (1 - \cos 2t - \cos^2 2t + \cos^3 2t)\,dt \qquad \text{Multiply terms.}$$

$$= \frac{3}{2}\left[\int_0^{\pi/2} (1 - \cos 2t)\,dt - \int_0^{\pi/2} \cos^2 2t\,dt + \int_0^{\pi/2} \cos^3 2t\,dt\right]$$

$$= \frac{3}{2}\left[\left(t - \frac{1}{2}\sin 2t\right) - \frac{1}{2}\left(t + \frac{1}{4}\sin 2t\right) + \frac{1}{2}\left(\sin 2t - \frac{1}{3}\sin^3 2t\right)\right]_0^{\pi/2} \qquad \text{Section 8.3, Example 3}$$

$$= \frac{3}{2}\left[\left(\frac{\pi}{2} - 0 - 0 - 0\right) - \frac{1}{2}\left(\frac{\pi}{2} + 0 - 0 - 0\right) + \frac{1}{2}(0 - 0 - 0 + 0)\right] \qquad \text{Evaluate.}$$

$$= \frac{3\pi}{8}.$$

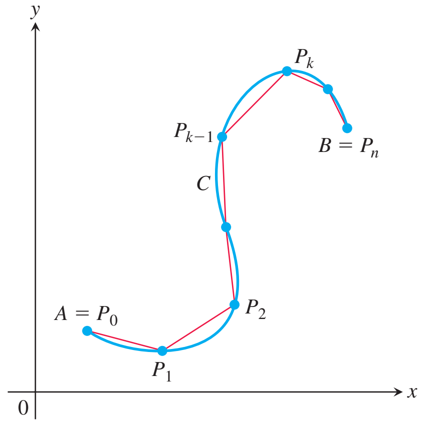


FIGURE 11.15 The length of the smooth curve C from A to B is approximated by the sum of the lengths of the polygonal path (straight-line segments) starting at $A = P_0$, then to P_1, and so on, ending at $B = P_n$.

Length of a Parametrically Defined Curve

Let C be a curve given parametrically by the equations

$$x = f(t) \quad \text{and} \quad y = g(t), \quad a \le t \le b.$$

We assume the functions f and g are **continuously differentiable** (meaning they have continuous first derivatives) on the interval $[a, b]$. We also assume that the derivatives $f'(t)$ and $g'(t)$ are not simultaneously zero, which prevents the curve C from having any corners or cusps. Such a curve is called a **smooth curve**. We subdivide the path (or arc) AB into n pieces at points $A = P_0, P_1, P_2, \dots, P_n = B$ (Figure 11.15). These points correspond to a partition of the interval $[a, b]$ by $a = t_0 < t_1 < t_2 < \cdots < t_n = b$, where $P_k = (f(t_k), g(t_k))$. Join successive points of this subdivision by straight-line segments (Figure 11.15). A representative line segment has length

$$L_k = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$$
$$= \sqrt{[f(t_k) - f(t_{k-1})]^2 + [g(t_k) - g(t_{k-1})]^2}$$

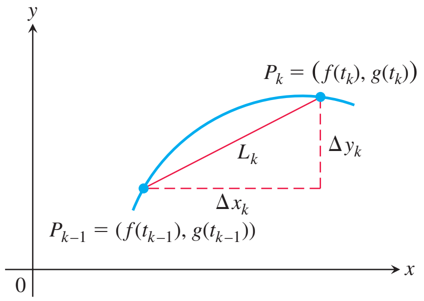


FIGURE 11.16 The arc $P_{k-1}P_k$ is approximated by the straight-line segment shown here, which has length $L_k = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$.

(see Figure 11.16). If Δt_k is small, the length L_k is approximately the length of arc $P_{k-1}P_k$. By the Mean Value Theorem there are numbers t_k^* and t_k^{**} in $[t_{k-1}, t_k]$ such that

$$\Delta x_k = f(t_k) - f(t_{k-1}) = f'(t_k^*)\,\Delta t_k,$$
$$\Delta y_k = g(t_k) - g(t_{k-1}) = g'(t_k^{**})\,\Delta t_k.$$

Assuming the path from A to B is traversed exactly once as t increases from $t = a$ to $t = b$, with no doubling back or retracing, an approximation to the (yet to be defined) "length" of the curve AB is the sum of all the lengths L_k:

$$\begin{aligned}\sum_{k=1}^{n} L_k &= \sum_{k=1}^{n} \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2} \\ &= \sum_{k=1}^{n} \sqrt{[f'(t_k^{*})]^2 + [g'(t_k^{**})]^2}\, \Delta t_k.\end{aligned}$$

Although this last sum on the right is not exactly a Riemann sum (because f' and g' are evaluated at different points), it can be shown that its limit, as the norm of the partition tends to zero and the number of segments $n \to \infty$, is the definite integral

$$\lim_{||P|| \to 0} \sum_{k=1}^{n} \sqrt{[f'(t_k^{*})]^2 + [g'(t_k^{**})]^2}\, \Delta t_k = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt.$$

Therefore, it is reasonable to define the length of the curve from A to B as this integral.

DEFINITION If a curve C is defined parametrically by $x = f(t)$ and $y = g(t)$, $a \le t \le b$, where f' and g' are continuous and not simultaneously zero on $[a, b]$, and C is traversed exactly once as t increases from $t = a$ to $t = b$, then **the length of C** is the definite integral

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt.$$

A smooth curve C does not double back or reverse the direction of motion over the time interval $[a, b]$ since $(f')^2 + (g')^2 > 0$ throughout the interval. At a point where a curve does start to double back on itself, either the curve fails to be differentiable or both derivatives must simultaneously equal zero. We will examine this phenomenon in Chapter 13, where we study tangent vectors to curves.

If $x = f(t)$ and $y = g(t)$, then using the Leibniz notation we have the following result for arc length:

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt. \tag{3}$$

If there are two different parametrizations for a curve C whose length we want to find, it does not matter which one we use. However, the parametrization we choose must meet the conditions stated in the definition of the length of C (see Exercise 41 for an example).

EXAMPLE 4 Using the definition, find the length of the circle of radius r defined parametrically by

$$x = r\cos t \qquad \text{and} \qquad y = r\sin t, \qquad 0 \le t \le 2\pi.$$

Solution As t varies from 0 to 2π, the circle is traversed exactly once, so the circumference is

$$L = \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

We find

$$\frac{dx}{dt} = -r \sin t, \qquad \frac{dy}{dt} = r \cos t$$

and

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = r^2(\sin^2 t + \cos^2 t) = r^2.$$

So

$$L = \int_0^{2\pi} \sqrt{r^2}\, dt = r\left[t\right]_0^{2\pi} = 2\pi r.$$ ■

EXAMPLE 5 Find the length of the astroid (Figure 11.14)

$$x = \cos^3 t, \qquad y = \sin^3 t, \qquad 0 \le t \le 2\pi.$$

Solution Because of the curve's symmetry with respect to the coordinate axes, its length is four times the length of the first-quadrant portion. We have

$$x = \cos^3 t, \qquad y = \sin^3 t$$

$$\left(\frac{dx}{dt}\right)^2 = \left[3\cos^2 t(-\sin t)\right]^2 = 9\cos^4 t \sin^2 t$$

$$\left(\frac{dy}{dt}\right)^2 = \left[3\sin^2 t(\cos t)\right]^2 = 9\sin^4 t \cos^2 t$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{9\cos^2 t \sin^2 t \underbrace{(\cos^2 t + \sin^2 t)}_{1}}$$

$$= \sqrt{9\cos^2 t \sin^2 t}$$

$$= 3\left|\cos t \sin t\right| \qquad \text{$\cos t \sin t \ge 0$ for $0 \le t \le \pi/2$}$$

$$= 3\cos t \sin t.$$

Therefore,

$$\text{Length of first-quadrant portion} = \int_0^{\pi/2} 3\cos t \sin t\, dt$$

$$= \frac{3}{2}\int_0^{\pi/2} \sin 2t\, dt \qquad \text{$\cos t \sin t = (1/2)\sin 2t$}$$

$$= -\frac{3}{4}\cos 2t\Big]_0^{\pi/2} = \frac{3}{2}.$$

The length of the astroid is four times this: $4(3/2) = 6$. ■

EXAMPLE 6 Find the perimeter of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

Solution Parametrically, we represent the ellipse by the equations $x = a \sin t$ and $y = b \cos t$, $a > b$ and $0 \le t \le 2\pi$. Then,

$$\begin{aligned}\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 &= a^2\cos^2 t + b^2\sin^2 t \\ &= a^2 - (a^2 - b^2)\sin^2 t \\ &= a^2[1 - e^2\sin^2 t]\end{aligned}$$

$e = 1 - \frac{b^2}{a^2}$, not the number 2.71828 . . .

From Equation (3), the perimeter is given by

$$P = 4a\int_0^{\pi/2}\sqrt{1 - e^2\sin^2 t}\,dt.$$

(We investigate the meaning of e in Section 11.7.) The integral for P is nonelementary and known as the *complete elliptic integral of the second kind.* We can compute its value to within any degree of accuracy using infinite series in the following way. From the binomial expansion for $\sqrt{1 - x}$ in Section 10.10, we have

$$\sqrt{1 - e^2\sin^2 t} = 1 - \frac{1}{2}e^2\sin^2 t - \frac{1}{2\cdot 4}e^4\sin^4 t - \cdots, \qquad |e\sin t| \le e < 1$$

Then to each term in this last expression we apply the integral Formula 157 (at the back of the book) for $\int_0^{\pi/2}\sin^n t\,dt$ when n is even, giving the perimeter

$$\begin{aligned}P &= 4a\int_0^{\pi/2}\sqrt{1 - e^2\sin^2 t}\,dt \\ &= 4a\left[\frac{\pi}{2} - \left(\frac{1}{2}e^2\right)\left(\frac{1}{2}\cdot\frac{\pi}{2}\right) - \left(\frac{1}{2\cdot 4}e^4\right)\left(\frac{1\cdot 3}{2\cdot 4}\cdot\frac{\pi}{2}\right) - \left(\frac{1\cdot 3}{2\cdot 4\cdot 6}e^6\right)\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{\pi}{2}\right) - \cdots\right] \\ &= 2\pi a\left[1 - \left(\frac{1}{2}\right)^2 e^2 - \left(\frac{1\cdot 3}{2\cdot 4}\right)^2\frac{e^4}{3} - \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2\frac{e^6}{5} - \cdots\right].\end{aligned}$$

Since $e < 1$, the series on the right-hand side converges by comparison with the geometric series $\sum_{n=1}^{\infty}(e^2)^n$. ■

Length of a Curve $y = f(x)$

HISTORICAL BIOGRAPHY
Gregory St. Vincent
(1584–1667)

The length formula in Section 6.3 is a special case of Equation (3). Given a continuously differentiable function $y = f(x)$, $a \le x \le b$, we can assign $x = t$ as a parameter. The graph of the function f is then the curve C defined parametrically by

$$x = t \quad \text{and} \quad y = f(t), \qquad a \le t \le b,$$

a special case of what we considered before. Then,

$$\frac{dx}{dt} = 1 \quad \text{and} \quad \frac{dy}{dt} = f'(t).$$

From Equation (1), we have

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = f'(t),$$

giving

$$\begin{aligned}\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 &= 1 + [f'(t)]^2 \\ &= 1 + [f'(x)]^2. \qquad t = x\end{aligned}$$

Substitution into Equation (3) gives the arc length formula for the graph of $y = f(x)$, in agreement with Equation (3) in Section 6.3.

The Arc Length Differential

Consistent with our discussion in Section 6.3, we can define the arc length function for a parametrically defined curve $x = f(t)$ and $y = g(t)$, $a \le t \le b$, by

$$s(t) = \int_a^t \sqrt{[f'(z)]^2 + [g'(z)]^2}\, dz.$$

Then, by the Fundamental Theorem of Calculus,

$$\frac{ds}{dt} = \sqrt{[f'(t)]^2 + [g'(t)]^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}.$$

The differential of arc length is

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt. \tag{4}$$

Equation (4) is often abbreviated to

$$ds = \sqrt{dx^2 + dy^2}.$$

Just as in Section 6.3, we can integrate the differential ds between appropriate limits to find the total length of a curve.

Here's an example where we use the arc length formula to find the centroid of an arc.

EXAMPLE 7 Find the centroid of the first-quadrant arc of the astroid in Example 5.

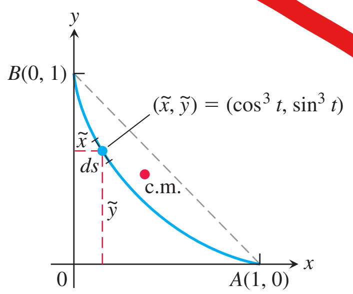


FIGURE 11.17 The centroid (c.m.) of the astroid arc in Example 7.

Solution We take the curve's density to be $\delta = 1$ and calculate the curve's mass and moments about the coordinate axes as we did in Section 6.6.

The distribution of mass is symmetric about the line $y = x$, so $\bar{x} = \bar{y}$. A typical segment of the curve (Figure 11.17) has mass

$$dm = 1 \cdot ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = 3 \cos t \sin t\, dt.$$ From Example 5

The curve's mass is

$$M = \int_0^{\pi/2} dm = \int_0^{\pi/2} 3 \cos t \sin t\, dt = \frac{3}{2}.$$ Again from Example 5

The curve's moment about the x-axis is

$$M_x = \int \tilde{y}\, dm = \int_0^{\pi/2} \sin^3 t \cdot 3 \cos t \sin t\, dt$$

$$= 3\int_0^{\pi/2} \sin^4 t \cos t\, dt = 3 \cdot \frac{\sin^5 t}{5}\Bigg]_0^{\pi/2} = \frac{3}{5}.$$

It follows that

$$\bar{y} = \frac{M_x}{M} = \frac{3/5}{3/2} = \frac{2}{5}.$$

The centroid is the point $(2/5, 2/5)$. ■

EXAMPLE 8 Find the time T_c it takes for a frictionless bead to slide along the cycloid $x = a(t - \sin t)$, $y = a(1 - \cos t)$ from $t = 0$ to $t = \pi$ (see Figure 11.12).

Solution From Equation (3) in Section 11.1, we want to find the time

$$T_c = \int_{t=0}^{t=\pi} \frac{ds}{\sqrt{2gy}}$$

for ds and y expressed parametrically in terms of the parameter t. For the cycloid, $dx/dt = a(1 - \cos t)$ and $dy/dt = a \sin t$, so

$$\begin{aligned} ds &= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\ &= \sqrt{a^2 (1 - 2\cos t + \cos^2 t + \sin^2 t)}\, dt \\ &= \sqrt{a^2 (2 - 2\cos t)}\, dt. \end{aligned}$$

Substituting for ds and y in the integrand, it follows that

$$\begin{aligned} T_c &= \int_0^{\pi} \sqrt{\frac{a^2(2 - 2\cos t)}{2ga(1 - \cos t)}}\, dt \\ &= \int_0^{\pi} \sqrt{\frac{a}{g}}\, dt = \pi\sqrt{\frac{a}{g}}, \end{aligned}$$

$y = a(1 - \cos t)$

which is the amount of time it takes the frictionless bead to slide down the cycloid to B after it is released from rest at O (see Figure 11.12). ■

Areas of Surfaces of Revolution

In Section 6.4 we found integral formulas for the area of a surface when a curve is revolved about a coordinate axis. Specifically, we found that the surface area is $S = \int 2\pi y\, ds$ for revolution about the x-axis, and $S = \int 2\pi x\, ds$ for revolution about the y-axis. If the curve is parametrized by the equations $x = f(t)$ and $y = g(t)$, $a \le t \le b$, where f and g are continuously differentiable and $(f')^2 + (g')^2 > 0$ on $[a, b]$, then the arc length differential ds is given by Equation (4). This observation leads to the following formulas for area of surfaces of revolution for smooth parametrized curves.

Area of Surface of Revolution for Parametrized Curves

If a smooth curve $x = f(t)$, $y = g(t)$, $a \le t \le b$, is traversed exactly once as t increases from a to b, then the areas of the surfaces generated by revolving the curve about the coordinate axes are as follows.

1. Revolution about the x-axis ($y \ge 0$):

$$S = \int_a^b 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \qquad (5)$$

2. Revolution about the y-axis ($x \ge 0$):

$$S = \int_a^b 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \qquad (6)$$

As with length, we can calculate surface area from any convenient parametrization that meets the stated criteria.

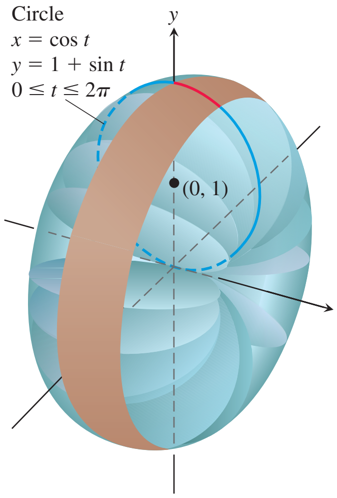


FIGURE 11.18 In Example 9 we calculate the area of the surface of revolution swept out by this parametrized curve.

EXAMPLE 9 The standard parametrization of the circle of radius 1 centered at the point (0, 1) in the xy-plane is

$$x = \cos t, \qquad y = 1 + \sin t, \qquad 0 \le t \le 2\pi.$$

Use this parametrization to find the area of the surface swept out by revolving the circle about the x-axis (Figure 11.18).

Solution We evaluate the formula

$$S = \int_a^b 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Eq. (5) for revolution about the x-axis; $y = 1 + \sin t \ge 0$

$$= \int_0^{2\pi} 2\pi(1 + \sin t)\underbrace{\sqrt{(-\sin t)^2 + (\cos t)^2}}_{1}\, dt$$

$$= 2\pi \int_0^{2\pi} (1 + \sin t)\, dt$$

$$= 2\pi \Big[t - \cos t \Big]_0^{2\pi} = 4\pi^2.$$

■

Exercises 11.2

Tangents to Parametrized Curves

In Exercises 1–14, find an equation for the line tangent to the curve at the point defined by the given value of t. Also, find the value of d^2y/dx^2 at this point.

1. $x = 2\cos t, \quad y = 2\sin t, \quad t = \pi/4$
2. $x = \sin 2\pi t, \quad y = \cos 2\pi t, \quad t = -1/6$
3. $x = 4\sin t, \quad y = 2\cos t, \quad t = \pi/4$
4. $x = \cos t, \quad y = \sqrt{3}\cos t, \quad t = 2\pi/3$
5. $x = t, \quad y = \sqrt{t}, \quad t = 1/4$
6. $x = \sec^2 t - 1, \quad y = \tan t, \quad t = -\pi/4$
7. $x = \sec t, \quad y = \tan t, \quad t = \pi/6$
8. $x = -\sqrt{t + 1}, \quad y = \sqrt{3t}, \quad t = 3$
9. $x = 2t^2 + 3, \quad y = t^4, \quad t = -1$
10. $x = 1/t, \quad y = -2 + \ln t, \quad t = 1$
11. $x = t - \sin t, \quad y = 1 - \cos t, \quad t = \pi/3$
12. $x = \cos t, \quad y = 1 + \sin t, \quad t = \pi/2$
13. $x = \dfrac{1}{t + 1}, \quad y = \dfrac{t}{t - 1}, \quad t = 2$
14. $x = t + e^t, \quad y = 1 - e^t, \quad t = 0$

Implicitly Defined Parametrizations

Assuming that the equations in Exercises 15–20 define x and y implicitly as differentiable functions $x = f(t)$, $y = g(t)$, find the slope of the curve $x = f(t)$, $y = g(t)$ at the given value of t.

15. $x^3 + 2t^2 = 9, \quad 2y^3 - 3t^2 = 4, \quad t = 2$
16. $x = \sqrt{5 - \sqrt{t}}, \quad y(t - 1) = \sqrt{t}, \quad t = 4$
17. $x + 2x^{3/2} = t^2 + t, \quad y\sqrt{t + 1} + 2t\sqrt{y} = 4, \quad t = 0$
18. $x \sin t + 2x = t, \quad t \sin t - 2t = y, \quad t = \pi$
19. $x = t^3 + t, \quad y + 2t^3 = 2x + t^2, \quad t = 1$
20. $t = \ln(x - t), \quad y = te^t, \quad t = 0$

Area

21. Find the area under one arch of the cycloid
$$x = a(t - \sin t), \quad y = a(1 - \cos t).$$
22. Find the area enclosed by the y-axis and the curve
$$x = t - t^2, \quad y = 1 + e^{-t}.$$
23. Find the area enclosed by the ellipse
$$x = a\cos t, \quad y = b\sin t, \quad 0 \le t \le 2\pi.$$
24. Find the area under $y = x^3$ over $[0, 1]$ using the following parametrizations.
   a. $x = t^2, \quad y = t^6$ **b.** $x = t^3, \quad y = t^9$

Lengths of Curves

Find the lengths of the curves in Exercises 25–30.

25. $x = \cos t, \quad y = t + \sin t, \quad 0 \le t \le \pi$
26. $x = t^3, \quad y = 3t^2/2, \quad 0 \le t \le \sqrt{3}$
27. $x = t^2/2, \quad y = (2t + 1)^{3/2}/3, \quad 0 \le t \le 4$
28. $x = (2t + 3)^{3/2}/3, \quad y = t + t^2/2, \quad 0 \le t \le 3$
29. $x = 8\cos t + 8t\sin t$, $y = 8\sin t - 8t\cos t$, $0 \le t \le \pi/2$
30. $x = \ln(\sec t + \tan t) - \sin t$, $y = \cos t, \quad 0 \le t \le \pi/3$

Surface Area

Find the areas of the surfaces generated by revolving the curves in Exercises 31–34 about the indicated axes.

31. $x = \cos t, \quad y = 2 + \sin t, \quad 0 \le t \le 2\pi; \quad x$-axis

32. $x = (2/3)t^{3/2}, \quad y = 2\sqrt{t}, \quad 0 \le t \le \sqrt{3}$; y-axis

33. $x = t + \sqrt{2}, \quad y = (t^2/2) + \sqrt{2}t, \quad -\sqrt{2} \le t \le \sqrt{2}$; y-axis

34. $x = \ln(\sec t + \tan t) - \sin t$, $y = \cos t$, $0 \le t \le \pi/3$; x-axis

35. A cone frustum The line segment joining the points (0, 1) and (2, 2) is revolved about the x-axis to generate a frustum of a cone. Find the surface area of the frustum using the parametrization $x = 2t$, $y = t + 1$, $0 \le t \le 1$. Check your result with the geometry formula: Area $= \pi(r_1 + r_2)$(slant height).

36. A cone The line segment joining the origin to the point (h, r) is revolved about the x-axis to generate a cone of height h and base radius r. Find the cone's surface area with the parametric equations $x = ht$, $y = rt$, $0 \le t \le 1$. Check your result with the geometry formula: Area $= \pi r$(slant height).

Centroids

37. Find the coordinates of the centroid of the curve

$$x = \cos t + t \sin t, \quad y = \sin t - t \cos t, \quad 0 \le t \le \pi/2.$$

38. Find the coordinates of the centroid of the curve

$$x = e^t \cos t, \quad y = e^t \sin t, \quad 0 \le t \le \pi.$$

39. Find the coordinates of the centroid of the curve

$$x = \cos t, \quad y = t + \sin t, \quad 0 \le t \le \pi.$$

T **40.** Most centroid calculations for curves are done with a calculator or computer that has an integral evaluation program. As a case in point, find, to the nearest hundredth, the coordinates of the centroid of the curve

$$x = t^3, \quad y = 3t^2/2, \quad 0 \le t \le \sqrt{3}.$$

Theory and Examples

41. Length is independent of parametrization To illustrate the fact that the numbers we get for length do not depend on the way we parametrize our curves (except for the mild restrictions preventing doubling back mentioned earlier), calculate the length of the semicircle $y = \sqrt{1 - x^2}$ with these two different parametrizations:

a. $x = \cos 2t, \quad y = \sin 2t, \quad 0 \le t \le \pi/2.$

b. $x = \sin \pi t, \quad y = \cos \pi t, \quad -1/2 \le t \le 1/2.$

42. a. Show that the Cartesian formula

$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

for the length of the curve $x = g(y)$, $c \le y \le d$ (Section 6.3, Equation 4), is a special case of the parametric length formula

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

Use this result to find the length of each curve.

b. $x = y^{3/2}, \quad 0 \le y \le 4/3$

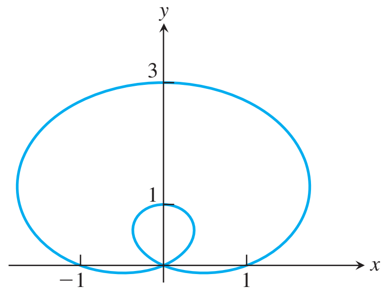
c. $x = \frac{3}{2}y^{2/3}, \quad 0 \le y \le 1$

43. The curve with parametric equations

$$x = (1 + 2\sin\theta)\cos\theta, \quad y = (1 + 2\sin\theta)\sin\theta$$

is called a *limaçon* and is shown in the accompanying figure. Find the points (x, y) and the slopes of the tangent lines at these points for

a. $\theta = 0$. **b.** $\theta = \pi/2$. **c.** $\theta = 4\pi/3$.

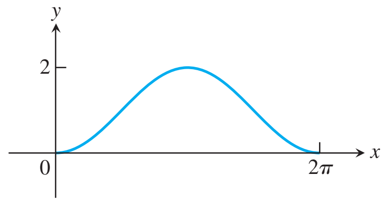


44. The curve with parametric equations

$$x = t, \quad y = 1 - \cos t, \quad 0 \le t \le 2\pi$$

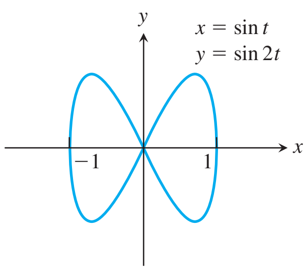
is called a *sinusoid* and is shown in the accompanying figure. Find the point (x, y) where the slope of the tangent line is

a. largest. **b.** smallest.

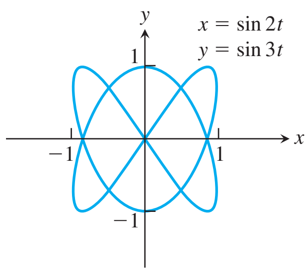


T The curves in Exercises 45 and 46 are called *Bowditch curves* or *Lissajous figures*. In each case, find the point in the interior of the first quadrant where the tangent to the curve is horizontal, and find the equations of the two tangents at the origin.

45.



46.



47. Cycloid

a. Find the length of one arch of the cycloid

$$x = a(t - \sin t), \quad y = a(1 - \cos t).$$

b. Find the area of the surface generated by revolving one arch of the cycloid in part (a) about the x-axis for $a = 1$.

48. Volume Find the volume swept out by revolving the region bounded by the x-axis and one arch of the cycloid

$$x = t - \sin t, \quad y = 1 - \cos t$$

about the x-axis.

COMPUTER EXPLORATIONS

In Exercises 49–52, use a CAS to perform the following steps for the given curve over the closed interval.

a. Plot the curve together with the polygonal path approximations for $n = 2, 4, 8$ partition points over the interval. (See Figure 11.15.)

b. Find the corresponding approximation to the length of the curve by summing the lengths of the line segments.

c. Evaluate the length of the curve using an integral. Compare your approximations for $n = 2, 4, 8$ with the actual length given by the integral. How does the actual length compare with the approximations as n increases? Explain your answer.

49. $x = \frac{1}{3}t^3, \quad y = \frac{1}{2}t^2, \quad 0 \le t \le 1$

50. $x = 2t^3 - 16t^2 + 25t + 5, \quad y = t^2 + t - 3, \quad 0 \le t \le 6$

51. $x = t - \cos t, \quad y = 1 + \sin t, \quad -\pi \le t \le \pi$

52. $x = e^t \cos t, \quad y = e^t \sin t, \quad 0 \le t \le \pi$

11.3 Polar Coordinates

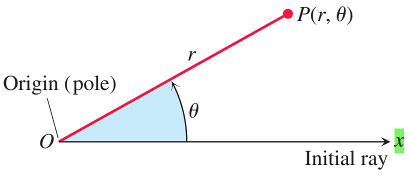


FIGURE 11.19 To define polar coordinates for the plane, we start with an origin, called the pole, and an initial ray.

In this section we study polar coordinates and their relation to Cartesian coordinates. You will see that polar coordinates are very useful for calculating many multiple integrals studied in Chapter 15. They are also useful in describing the paths of planets and satellites.

Definition of Polar Coordinates

To define polar coordinates, we first fix an **origin** O (called the **pole**) and an **initial ray** from O (Figure 11.19). Usually the positive x-axis is chosen as the initial ray. Then each point P can be located by assigning to it a **polar coordinate pair** (r, θ) in which r gives the directed distance from O to P and θ gives the directed angle from the initial ray to ray OP. So we label the point P as

$$P(r, \theta)$$

Directed distance from O to P — Directed angle from initial ray to OP

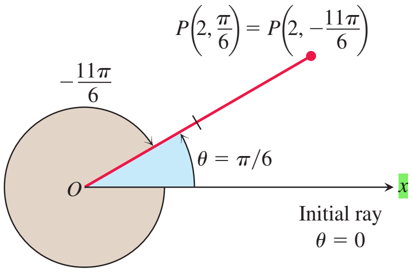


FIGURE 11.20 Polar coordinates are not unique.

As in trigonometry, θ is positive when measured counterclockwise and negative when measured clockwise. The angle associated with a given point is not unique. While a point in the plane has just one pair of Cartesian coordinates, it has infinitely many pairs of polar coordinates. For instance, the point 2 units from the origin along the ray $\theta = \pi/6$ has polar coordinates $r = 2,\ \theta = \pi/6$. It also has coordinates $r = 2, \theta = -11\pi/6$ (Figure 11.20). In some situations we allow r to be negative. That is why we use directed distance in defining $P(r, \theta)$. The point $P(2, 7\pi/6)$ can be reached by turning $7\pi/6$ radians counterclockwise from the initial ray and going forward 2 units (Figure 11.21). It can also be reached by turning $\pi/6$ radians counterclockwise from the initial ray and going *backward* 2 units. So the point also has polar coordinates $r = -2, \theta = \pi/6$.

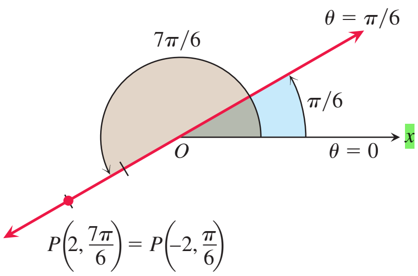


FIGURE 11.21 Polar coordinates can have negative r-values.

EXAMPLE 1 Find all the polar coordinates of the point $P(2, \pi/6)$.

Solution We sketch the initial ray of the coordinate system, draw the ray from the origin that makes an angle of $\pi/6$ radians with the initial ray, and mark the point $(2, \pi/6)$ (Figure 11.22). We then find the angles for the other coordinate pairs of P in which $r = 2$ and $r = -2$.

For $r = 2$, the complete list of angles is

$$\frac{\pi}{6}, \quad \frac{\pi}{6} \pm 2\pi, \quad \frac{\pi}{6} \pm 4\pi, \quad \frac{\pi}{6} \pm 6\pi, \ldots.$$

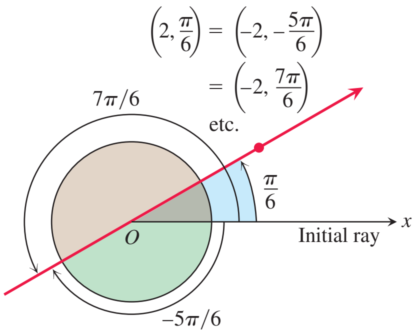


FIGURE 11.22 The point $P(2, \pi/6)$ has infinitely many polar coordinate pairs (Example 1).

For $r = -2$, the angles are

$$-\frac{5\pi}{6}, \quad -\frac{5\pi}{6} \pm 2\pi, \quad -\frac{5\pi}{6} \pm 4\pi, \quad -\frac{5\pi}{6} \pm 6\pi, \ldots .$$

The corresponding coordinate pairs of P are

$$\left(2, \frac{\pi}{6} + 2n\pi\right), \qquad n = 0, \pm 1, \pm 2, \ldots$$

and

$$\left(-2, -\frac{5\pi}{6} + 2n\pi\right), \qquad n = 0, \pm 1, \pm 2, \ldots .$$

When $n = 0$, the formulas give $(2, \pi/6)$ and $(-2, -5\pi/6)$. When $n = 1$, they give $(2, 13\pi/6)$ and $(-2, 7\pi/6)$, and so on. ■

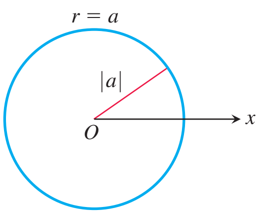


FIGURE 11.23 The polar equation for a circle is $r = a$.

Polar Equations and Graphs

If we hold r fixed at a constant value $r = a \neq 0$, the point $P(r, \theta)$ will lie $|a|$ units from the origin O. As θ varies over any interval of length 2π, P then traces a circle of radius $|a|$ centered at O (Figure 11.23).

If we hold θ fixed at a constant value $\theta = \theta_0$ and let r vary between $-\infty$ and ∞, the point $P(r, \theta)$ traces the line through O that makes an angle of measure θ_0 with the initial ray. (See Figure 11.21 for an example.)

EXAMPLE 2 A circle or line can have more than one polar equation.

(a) $r = 1$ and $r = -1$ are equations for the circle of radius 1 centered at O.

(b) $\theta = \pi/6$, $\theta = 7\pi/6$, and $\theta = -5\pi/6$ are equations for the line in Figure 11.22. ■

Equations of the form $r = a$ and $\theta = \theta_0$ can be combined to define regions, segments, and rays.

EXAMPLE 3 Graph the sets of points whose polar coordinates satisfy the following conditions.

(a) $1 \le r \le 2 \quad$ and $\quad 0 \le \theta \le \dfrac{\pi}{2}$

(b) $-3 \le r \le 2 \quad$ and $\quad \theta = \dfrac{\pi}{4}$

(c) $\dfrac{2\pi}{3} \le \theta \le \dfrac{5\pi}{6} \quad$ (no restriction on r)

Solution The graphs are shown in Figure 11.24. ■

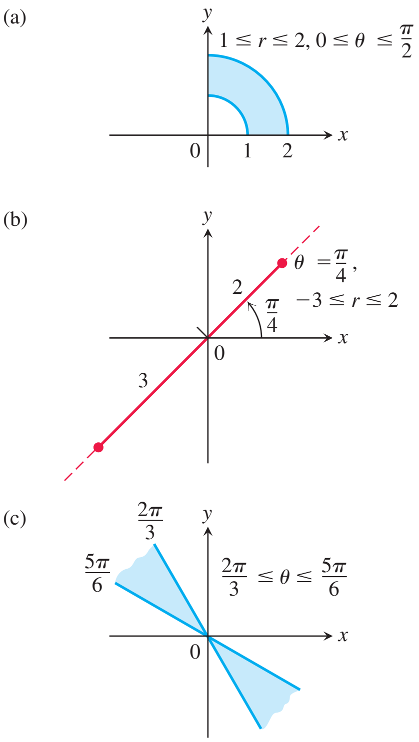


FIGURE 11.24 The graphs of typical inequalities in r and θ (Example 3).

Relating Polar and Cartesian Coordinates

When we use both polar and Cartesian coordinates in a plane, we place the two origins together and take the initial polar ray as the positive x-axis. The ray $\theta = \pi/2, r > 0$,

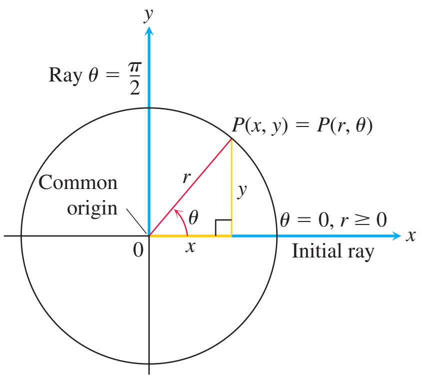


FIGURE 11.25 The usual way to relate polar and Cartesian coordinates.

becomes the positive y-axis (Figure 11.25). The two coordinate systems are then related by the following equations.

> **Equations Relating Polar and Cartesian Coordinates**
>
> $$x = r\cos\theta, \qquad y = r\sin\theta, \qquad r^2 = x^2 + y^2, \qquad \tan\theta = \frac{y}{x}$$

The first two of these equations uniquely determine the Cartesian coordinates x and y given the polar coordinates r and θ. On the other hand, if x and y are given, the third equation gives two possible choices for r (a positive and a negative value). For each $(x, y) \neq (0, 0)$, there is a unique $\theta \in [0, 2\pi)$ satisfying the first two equations, each then giving a polar coordinate representation of the Cartesian point (x, y). The other polar coordinate representations for the point can be determined from these two, as in Example 1.

EXAMPLE 4 Here are some plane curves expressed in terms of both polar coordinate and Cartesian coordinate equations.

Polar equation	Cartesian equivalent
$r\cos\theta = 2$	$x = 2$
$r^2\cos\theta\sin\theta = 4$	$xy = 4$
$r^2\cos^2\theta - r^2\sin^2\theta = 1$	$x^2 - y^2 = 1$
$r = 1 + 2r\cos\theta$	$y^2 - 3x^2 - 4x - 1 = 0$
$r = 1 - \cos\theta$	$x^4 + y^4 + 2x^2y^2 + 2x^3 + 2xy^2 - y^2 = 0$

Some curves are more simply expressed with polar coordinates; others are not. ■

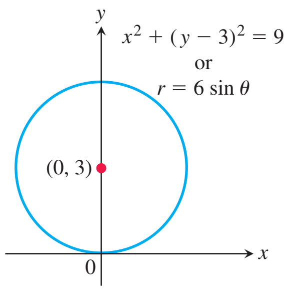


FIGURE 11.26 The circle in Example 5.

EXAMPLE 5 Find a polar equation for the circle $x^2 + (y - 3)^2 = 9$ (Figure 11.26).

Solution We apply the equations relating polar and Cartesian coordinates:

$$\begin{aligned}
x^2 + (y-3)^2 &= 9 \\
x^2 + y^2 - 6y + 9 &= 9 && \text{Expand } (y-3)^2. \\
x^2 + y^2 - 6y &= 0 && \text{Cancelation} \\
r^2 - 6r\sin\theta &= 0 && x^2 + y^2 = r^2,\ y = r\sin\theta \\
r = 0 \quad \text{or} \quad r - 6\sin\theta &= 0 \\
r &= 6\sin\theta && \text{Includes both possibilities}
\end{aligned}$$

■

EXAMPLE 6 Replace the following polar equations by equivalent Cartesian equations and identify their graphs.

(a) $r\cos\theta = -4$

(b) $r^2 = 4r\cos\theta$

(c) $r = \dfrac{4}{2\cos\theta - \sin\theta}$

Solution We use the substitutions $r\cos\theta = x$, $r\sin\theta = y$, and $r^2 = x^2 + y^2$.

(a) $r\cos\theta = -4$

The Cartesian equation:
$$\begin{aligned} r\cos\theta &= -4 \\ x &= -4 && \text{Substitution} \end{aligned}$$

The graph: Vertical line through $x = -4$ on the x-axis

(b) $r^2 = 4r\cos\theta$

The Cartesian equation:

$$r^2 = 4r\cos\theta$$
$$x^2 + y^2 = 4x \quad \text{Substitution}$$
$$x^2 - 4x + y^2 = 0$$
$$x^2 - 4x + 4 + y^2 = 4 \quad \text{Completing the square}$$
$$(x-2)^2 + y^2 = 4 \quad \text{Factoring}$$

The graph: Circle, radius 2, center $(h, k) = (2, 0)$

(c) $r = \dfrac{4}{2\cos\theta - \sin\theta}$

The Cartesian equation:

$$r(2\cos\theta - \sin\theta) = 4$$
$$2r\cos\theta - r\sin\theta = 4 \quad \text{Multiplying by } r$$
$$2x - y = 4 \quad \text{Substitution}$$
$$y = 2x - 4 \quad \text{Solve for } y.$$

The graph: Line, slope $m = 2$, y-intercept $b = -4$ ■

Exercises 11.3

Polar Coordinates

1. Which polar coordinate pairs label the same point?

a. $(3, 0)$ **b.** $(-3, 0)$ **c.** $(2, 2\pi/3)$
d. $(2, 7\pi/3)$ **e.** $(-3, \pi)$ **f.** $(2, \pi/3)$
g. $(-3, 2\pi)$ **h.** $(-2, -\pi/3)$

2. Which polar coordinate pairs label the same point?

a. $(-2, \pi/3)$ **b.** $(2, -\pi/3)$ **c.** (r, θ)
d. $(r, \theta + \pi)$ **e.** $(-r, \theta)$ **f.** $(2, -2\pi/3)$
g. $(-r, \theta + \pi)$ **h.** $(-2, 2\pi/3)$

3. Plot the following points (given in polar coordinates). Then find all the polar coordinates of each point.

a. $(2, \pi/2)$ **b.** $(2, 0)$
c. $(-2, \pi/2)$ **d.** $(-2, 0)$

4. Plot the following points (given in polar coordinates). Then find all the polar coordinates of each point.

a. $(3, \pi/4)$ **b.** $(-3, \pi/4)$
c. $(3, -\pi/4)$ **d.** $(-3, -\pi/4)$

Polar to Cartesian Coordinates

5. Find the Cartesian coordinates of the points in Exercise 1.

6. Find the Cartesian coordinates of the following points (given in polar coordinates).

a. $(\sqrt{2}, \pi/4)$ **b.** $(1, 0)$
c. $(0, \pi/2)$ **d.** $(-\sqrt{2}, \pi/4)$
e. $(-3, 5\pi/6)$ **f.** $(5, \tan^{-1}(4/3))$
g. $(-1, 7\pi)$ **h.** $(2\sqrt{3}, 2\pi/3)$

Cartesian to Polar Coordinates

7. Find the polar coordinates, $0 \le \theta < 2\pi$ and $r \ge 0$, of the following points given in Cartesian coordinates.

a. $(1, 1)$ **b.** $(-3, 0)$
c. $(\sqrt{3}, -1)$ **d.** $(-3, 4)$

8. Find the polar coordinates, $-\pi \le \theta < \pi$ and $r \ge 0$, of the following points given in Cartesian coordinates.

a. $(-2, -2)$ **b.** $(0, 3)$
c. $(-\sqrt{3}, 1)$ **d.** $(5, -12)$

9. Find the polar coordinates, $0 \le \theta < 2\pi$ and $r \le 0$, of the following points given in Cartesian coordinates.

a. $(3, 3)$ **b.** $(-1, 0)$
c. $(-1, \sqrt{3})$ **d.** $(4, -3)$

10. Find the polar coordinates, $-\pi \le \theta < 2\pi$ and $r \le 0$, of the following points given in Cartesian coordinates.

a. $(-2, 0)$ **b.** $(1, 0)$
c. $(0, -3)$ **d.** $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

Graphing Sets of Polar Coordinate Points

Graph the sets of points whose polar coordinates satisfy the equations and inequalities in Exercises 11–26.

11. $r = 2$ **12.** $0 \le r \le 2$
13. $r \ge 1$ **14.** $1 \le r \le 2$

15. $0 \le \theta \le \pi/6, \quad r \ge 0$
16. $\theta = 2\pi/3, \quad r \le -2$
17. $\theta = \pi/3, \quad -1 \le r \le 3$
18. $\theta = 11\pi/4, \quad r \ge -1$
19. $\theta = \pi/2, \quad r \ge 0$
20. $\theta = \pi/2, \quad r \le 0$
21. $0 \le \theta \le \pi, \quad r = 1$
22. $0 \le \theta \le \pi, \quad r = -1$
23. $\pi/4 \le \theta \le 3\pi/4, \quad 0 \le r \le 1$
24. $-\pi/4 \le \theta \le \pi/4, \quad -1 \le r \le 1$
25. $-\pi/2 \le \theta \le \pi/2, \quad 1 \le r \le 2$
26. $0 \le \theta \le \pi/2, \quad 1 \le |r| \le 2$

Polar to Cartesian Equations

Replace the polar equations in Exercises 27–52 with equivalent Cartesian equations. Then describe or identify the graph.

27. $r\cos\theta = 2$
28. $r\sin\theta = -1$
29. $r\sin\theta = 0$
30. $r\cos\theta = 0$
31. $r = 4\csc\theta$
32. $r = -3\sec\theta$
33. $r\cos\theta + r\sin\theta = 1$
34. $r\sin\theta = r\cos\theta$
35. $r^2 = 1$
36. $r^2 = 4r\sin\theta$
37. $r = \dfrac{5}{\sin\theta - 2\cos\theta}$
38. $r^2\sin 2\theta = 2$
39. $r = \cot\theta\csc\theta$
40. $r = 4\tan\theta\sec\theta$
41. $r = \csc\theta\, e^{r\cos\theta}$
42. $r\sin\theta = \ln r + \ln\cos\theta$
43. $r^2 + 2r^2\cos\theta\sin\theta = 1$
44. $\cos^2\theta = \sin^2\theta$
45. $r^2 = -4r\cos\theta$
46. $r^2 = -6r\sin\theta$
47. $r = 8\sin\theta$
48. $r = 3\cos\theta$
49. $r = 2\cos\theta + 2\sin\theta$
50. $r = 2\cos\theta - \sin\theta$
51. $r\sin\left(\theta + \dfrac{\pi}{6}\right) = 2$
52. $r\sin\left(\dfrac{2\pi}{3} - \theta\right) = 5$

Cartesian to Polar Equations

Replace the Cartesian equations in Exercises 53–66 with equivalent polar equations.

53. $x = 7$
54. $y = 1$
55. $x = y$
56. $x - y = 3$
57. $x^2 + y^2 = 4$
58. $x^2 - y^2 = 1$
59. $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$
60. $xy = 2$
61. $y^2 = 4x$
62. $x^2 + xy + y^2 = 1$
63. $x^2 + (y - 2)^2 = 4$
64. $(x - 5)^2 + y^2 = 25$
65. $(x - 3)^2 + (y + 1)^2 = 4$
66. $(x + 2)^2 + (y - 5)^2 = 16$
67. Find all polar coordinates of the origin.
68. **Vertical and horizontal lines**
    a. Show that every vertical line in the xy-plane has a polar equation of the form $r = a\sec\theta$.
    b. Find the analogous polar equation for horizontal lines in the xy-plane.

11.4 Graphing Polar Coordinate Equations

It is often helpful to graph an equation expressed in polar coordinates in the Cartesian xy-plane. This section describes some techniques for graphing these equations using symmetries and tangents to the graph.

Symmetry

Figure 11.27 illustrates the standard polar coordinate tests for symmetry. The following summary says how the symmetric points are related.

Symmetry Tests for Polar Graphs in the Cartesian xy-Plane

1. *Symmetry about the x-axis:* If the point (r, θ) lies on the graph, then the point $(r, -\theta)$ or $(-r, \pi - \theta)$ lies on the graph (Figure 11.27a).
2. *Symmetry about the y-axis:* If the point (r, θ) lies on the graph, then the point $(r, \pi - \theta)$ or $(-r, -\theta)$ lies on the graph (Figure 11.27b).
3. *Symmetry about the origin:* If the point (r, θ) lies on the graph, then the point $(-r, \theta)$ or $(r, \theta + \pi)$ lies on the graph (Figure 11.27c).

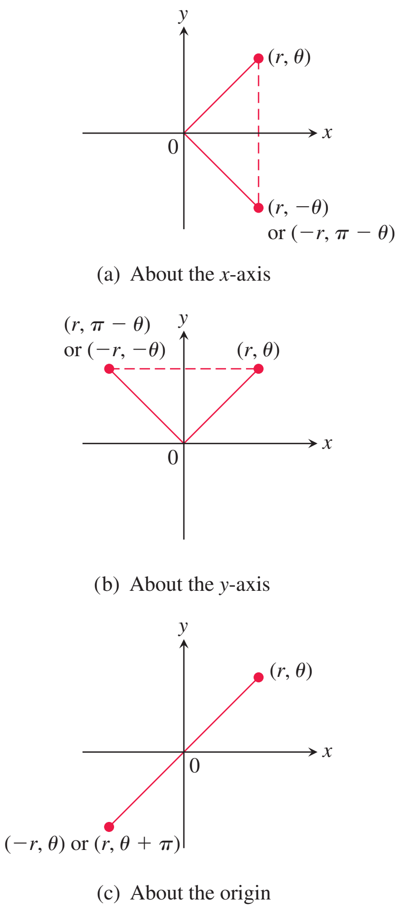


FIGURE 11.27 Three tests for symmetry in polar coordinates.

Slope

The slope of a polar curve $r = f(\theta)$ in the xy-plane is still given by dy/dx, which is not $r' = df/d\theta$. To see why, think of the graph of f as the graph of the parametric equations

$$x = r\cos\theta = f(\theta)\cos\theta, \qquad y = r\sin\theta = f(\theta)\sin\theta.$$

If f is a differentiable function of θ, then so are x and y and, when $dx/d\theta \neq 0$, we can calculate dy/dx from the parametric formula

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} \qquad \text{Section 11.2, Eq. (1) with } t = \theta$$

$$= \frac{\frac{d}{d\theta}(f(\theta)\cdot\sin\theta)}{\frac{d}{d\theta}(f(\theta)\cdot\cos\theta)}$$

$$= \frac{\frac{df}{d\theta}\sin\theta + f(\theta)\cos\theta}{\frac{df}{d\theta}\cos\theta - f(\theta)\sin\theta} \qquad \text{Product Rule for derivatives}$$

Therefore we see that dy/dx is not the same as $df/d\theta$.

Slope of the Curve $r = f(\theta)$ in the Cartesian xy-Plane

$$\left.\frac{dy}{dx}\right|_{(r,\theta)} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$$

provided $dx/d\theta \neq 0$ at (r, θ).

If the curve $r = f(\theta)$ passes through the origin at $\theta = \theta_0$, then $f(\theta_0) = 0$, and the slope equation gives

$$\left.\frac{dy}{dx}\right|_{(0,\theta_0)} = \frac{f'(\theta_0)\sin\theta_0}{f'(\theta_0)\cos\theta_0} = \tan\theta_0.$$

If the graph of $r = f(\theta)$ passes through the origin at the value $\theta = \theta_0$, the slope of the curve there is $\tan\theta_0$. The reason we say "slope at $(0, \theta_0)$" and not just "slope at the origin" is that a polar curve may pass through the origin (or any point) more than once, with different slopes at different θ-values. This is not the case in our first example, however.

EXAMPLE 1 Graph the curve $r = 1 - \cos\theta$ in the Cartesian xy-plane.

Solution The curve is symmetric about the x-axis because

$$(r, \theta) \text{ on the graph} \Rightarrow r = 1 - \cos\theta$$
$$\Rightarrow r = 1 - \cos(-\theta) \qquad \cos\theta = \cos(-\theta)$$
$$\Rightarrow (r, -\theta) \text{ on the graph.}$$

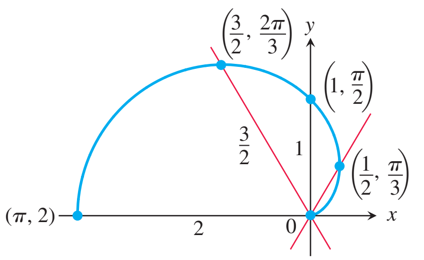
As θ increases from 0 to π, $\cos\theta$ decreases from 1 to -1, and $r = 1 - \cos\theta$ increases from a minimum value of 0 to a maximum value of 2. As θ continues on from π to 2π, $\cos\theta$ increases from -1 back to 1 and r decreases from 2 back to 0. The curve starts to repeat when $\theta = 2\pi$ because the cosine has period 2π.

The curve leaves the origin with slope $\tan(0) = 0$ and returns to the origin with slope $\tan(2\pi) = 0$.

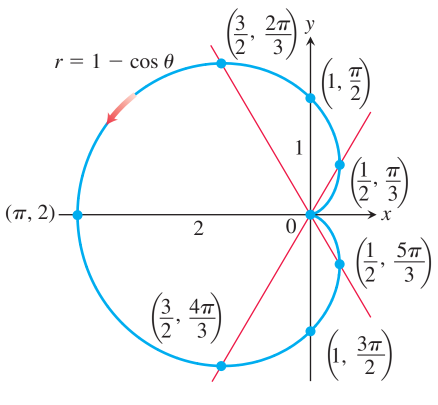
We make a table of values from $\theta = 0$ to $\theta = \pi$, plot the points, draw a smooth curve through them with a horizontal tangent at the origin, and reflect the curve across the x-axis to complete the graph (Figure 11.28). The curve is called a *cardioid* because of its heart shape. ■

θ	$r = 1 - \cos\theta$
0	0
$\frac{\pi}{3}$	$\frac{1}{2}$
$\frac{\pi}{2}$	1
$\frac{2\pi}{3}$	$\frac{3}{2}$
π	2

(a)



(b)



(c)

FIGURE 11.28 The steps in graphing the cardioid $r = 1 - \cos\theta$ (Example 1). The arrow shows the direction of increasing θ.

EXAMPLE 2 Graph the curve $r^2 = 4\cos\theta$ in the Cartesian xy-plane.

Solution The equation $r^2 = 4\cos\theta$ requires $\cos\theta \geq 0$, so we get the entire graph by running θ from $-\pi/2$ to $\pi/2$. The curve is symmetric about the x-axis because

$$
\begin{aligned}
(r, \theta) \text{ on the graph} &\Rightarrow r^2 = 4\cos\theta \\
&\Rightarrow r^2 = 4\cos(-\theta) \qquad \cos\theta = \cos(-\theta) \\
&\Rightarrow (r, -\theta) \text{ on the graph.}
\end{aligned}
$$

The curve is also symmetric about the origin because

$$
\begin{aligned}
(r, \theta) \text{ on the graph} &\Rightarrow r^2 = 4\cos\theta \\
&\Rightarrow (-r)^2 = 4\cos\theta \\
&\Rightarrow (-r, \theta) \text{ on the graph.}
\end{aligned}
$$

Together, these two symmetries imply symmetry about the y-axis.

The curve passes through the origin when $\theta = -\pi/2$ and $\theta = \pi/2$. It has a vertical tangent both times because $\tan\theta$ is infinite.

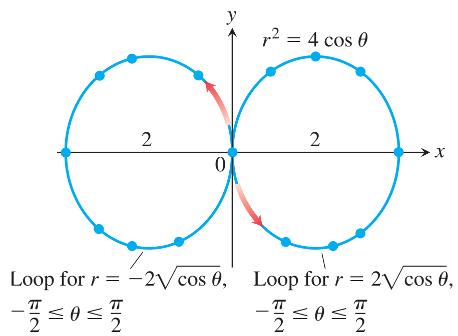
For each value of θ in the interval between $-\pi/2$ and $\pi/2$, the formula $r^2 = 4\cos\theta$ gives two values of r:

$$r = \pm 2\sqrt{\cos\theta}.$$

We make a short table of values, plot the corresponding points, and use information about symmetry and tangents to guide us in connecting the points with a smooth curve (Figure 11.29).

θ	$\cos\theta$	$r = \pm 2\sqrt{\cos\theta}$
0	1	± 2
$\pm\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	$\approx \pm 1.9$
$\pm\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\approx \pm 1.7$
$\pm\frac{\pi}{3}$	$\frac{1}{2}$	$\approx \pm 1.4$
$\pm\frac{\pi}{2}$	0	0

(a)



(b)

FIGURE 11.29 The graph of $r^2 = 4\cos\theta$. The arrows show the direction of increasing θ. The values of r in the table are rounded (Example 2). ■

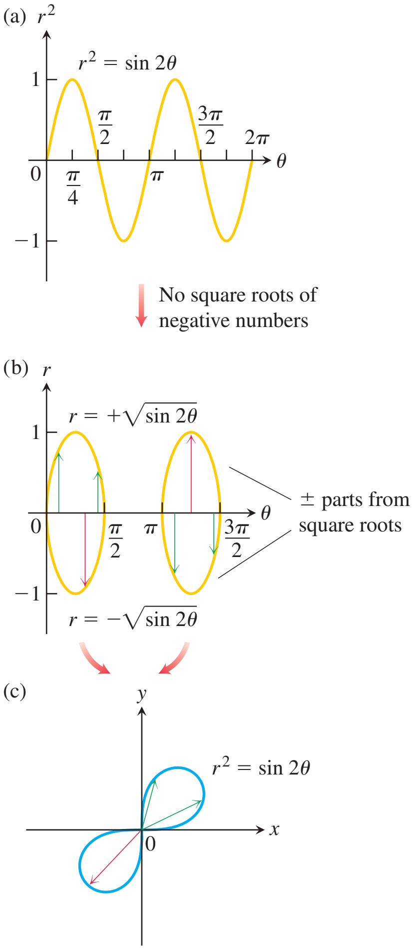


FIGURE 11.30 To plot $r = f(\theta)$ in the Cartesian $r\theta$-plane in (b), we first plot $r^2 = \sin 2\theta$ in the $r^2\theta$-plane in (a) and then ignore the values of θ for which $\sin 2\theta$ is negative. The radii from the sketch in (b) cover the polar graph of the lemniscate in (c) twice (Example 3).

Converting a Graph from the $r\theta$- to xy-Plane

One way to graph a polar equation $r = f(\theta)$ in the xy-plane is to make a table of (r, θ)-values, plot the corresponding points there, and connect them in order of increasing θ. This can work well if enough points have been plotted to reveal all the loops and dimples in the graph. Another method of graphing is to

1. first graph the function $r = f(\theta)$ in the *Cartesian* $r\theta$-plane,
2. then use that Cartesian graph as a "table" and guide to sketch the *polar* coordinate graph in the xy-plane.

This method is sometimes better than simple point plotting because the first Cartesian graph, even when hastily drawn, shows at a glance where r is positive, negative, and nonexistent, as well as where r is increasing and decreasing. Here's an example.

EXAMPLE 3 Graph the *lemniscate* curve $r^2 = \sin 2\theta$ in the Cartesian xy-plane.

Solution Here we begin by plotting r^2 (not r) as a function of θ in the Cartesian $r^2\theta$-plane. See Figure 11.30a. We pass from there to the graph of $r = \pm\sqrt{\sin 2\theta}$ in the $r\theta$-plane (Figure 11.30b), and then draw the polar graph (Figure 11.30c). The graph in Figure 11.30b "covers" the final polar graph in Figure 11.30c twice. We could have managed with either loop alone, with the two upper halves, or with the two lower halves. The double covering does no harm, however, and we actually learn a little more about the behavior of the function this way. ■

USING TECHNOLOGY Graphing Polar Curves Parametrically

For complicated polar curves we may need to use a graphing calculator or computer to graph the curve. If the device does not plot polar graphs directly, we can convert $r = f(\theta)$ into parametric form using the equations

$$x = r\cos\theta = f(\theta)\cos\theta, \qquad y = r\sin\theta = f(\theta)\sin\theta.$$

Then we use the device to draw a parametrized curve in the Cartesian xy-plane. It may be necessary to use the parameter t rather than θ for the graphing device.

Exercises 11.4

Symmetries and Polar Graphs

Identify the symmetries of the curves in Exercises 1–12. Then sketch the curves in the xy-plane.

1. $r = 1 + \cos\theta$ **2.** $r = 2 - 2\cos\theta$

3. $r = 1 - \sin\theta$ **4.** $r = 1 + \sin\theta$

5. $r = 2 + \sin\theta$ **6.** $r = 1 + 2\sin\theta$

7. $r = \sin(\theta/2)$ **8.** $r = \cos(\theta/2)$

9. $r^2 = \cos\theta$ **10.** $r^2 = \sin\theta$

11. $r^2 = -\sin\theta$ **12.** $r^2 = -\cos\theta$

Graph the lemniscates in Exercises 13–16. What symmetries do these curves have?

13. $r^2 = 4\cos 2\theta$ **14.** $r^2 = 4\sin 2\theta$

15. $r^2 = -\sin 2\theta$ **16.** $r^2 = -\cos 2\theta$

Slopes of Polar Curves in the xy-Plane

Find the slopes of the curves in Exercises 17–20 at the given points. Sketch the curves along with their tangents at these points.

17. Cardioid $r = -1 + \cos\theta$; $\theta = \pm\pi/2$

18. Cardioid $r = -1 + \sin\theta$; $\theta = 0, \pi$

19. Four-leaved rose $r = \sin 2\theta$; $\theta = \pm\pi/4, \pm 3\pi/4$

20. Four-leaved rose $r = \cos 2\theta$; $\theta = 0, \pm\pi/2, \pi$

Graphing Limaçons

Graph the limaçons in Exercises 21–24. Limaçon ("*lee*-ma-sahn") is Old French for "snail." You will understand the name when you graph the limaçons in Exercise 21. Equations for limaçons have the form $r = a \pm b\cos\theta$ or $r = a \pm b\sin\theta$. There are four basic shapes.

21. Limaçons with an inner loop

a. $r = \frac{1}{2} + \cos\theta$ **b.** $r = \frac{1}{2} + \sin\theta$

22. Cardioids

a. $r = 1 - \cos\theta$ **b.** $r = -1 + \sin\theta$

23. Dimpled limaçons

a. $r = \frac{3}{2} + \cos\theta$ **b.** $r = \frac{3}{2} - \sin\theta$

24. Oval limaçons

a. $r = 2 + \cos\theta$ **b.** $r = -2 + \sin\theta$

Graphing Polar Regions and Curves in the *xy*-Plane

25. Sketch the region defined by the inequalities $-1 \le r \le 2$ and $-\pi/2 \le \theta \le \pi/2$.

26. Sketch the region defined by the inequalities $0 \le r \le 2\sec\theta$ and $-\pi/4 \le \theta \le \pi/4$.

In Exercises 27 and 28, sketch the region defined by the inequality.

27. $0 \le r \le 2 - 2\cos\theta$ **28.** $0 \le r^2 \le \cos\theta$

T **29.** Which of the following has the same graph as $r = 1 - \cos\theta$?

a. $r = -1 - \cos\theta$ **b.** $r = 1 + \cos\theta$

Confirm your answer with algebra.

T **30.** Which of the following has the same graph as $r = \cos 2\theta$?

a. $r = -\sin(2\theta + \pi/2)$ **b.** $r = -\cos(\theta/2)$

Confirm your answer with algebra.

T **31. A rose within a rose** Graph the equation $r = 1 - 2\sin 3\theta$.

T **32. The nephroid of Freeth** Graph the nephroid of Freeth:

$$r = 1 + 2\sin\frac{\theta}{2}.$$

T **33. Roses** Graph the roses $r = \cos m\theta$ for $m = 1/3, 2, 3$, and 7.

T **34. Spirals** Polar coordinates are just the thing for defining spirals. Graph the following spirals.

a. $r = \theta$

b. $r = -\theta$

c. *A logarithmic spiral:* $r = e^{\theta/10}$

d. *A hyperbolic spiral:* $r = 8/\theta$

e. *An equilateral hyperbola:* $r = \pm 10/\sqrt{\theta}$

(Use different colors for the two branches.)

T **35.** Graph the equation $r = \sin\left(\frac{8}{7}\theta\right)$ for $0 \le \theta \le 14\pi$.

T **36.** Graph the equation

$$r = \sin^2(2.3\theta) + \cos^4(2.3\theta)$$

for $0 \le \theta \le 10\pi$.

11.5 Areas and Lengths in Polar Coordinates

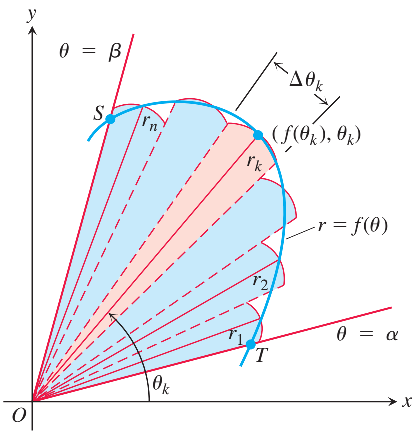


FIGURE 11.31 To derive a formula for the area of region *OTS*, we approximate the region with fan-shaped circular sectors.

This section shows how to calculate areas of plane regions and lengths of curves in polar coordinates. The defining ideas are the same as before, but the formulas are different in polar versus Cartesian coordinates.

Area in the Plane

The region *OTS* in Figure 11.31 is bounded by the rays $\theta = \alpha$ and $\theta = \beta$ and the curve $r = f(\theta)$. We approximate the region with n nonoverlapping fan-shaped circular sectors based on a partition P of angle *TOS*. The typical sector has radius $r_k = f(\theta_k)$ and central angle of radian measure $\Delta\theta_k$. Its area is $\Delta\theta_k/2\pi$ times the area of a circle of radius r_k, or

$$A_k = \frac{1}{2}r_k{}^2\,\Delta\theta_k = \frac{1}{2}\left(f(\theta_k)\right)^2\Delta\theta_k.$$

The area of region *OTS* is approximately

$$\sum_{k=1}^{n} A_k = \sum_{k=1}^{n}\frac{1}{2}\left(f(\theta_k)\right)^2\Delta\theta_k.$$

If f is continuous, we expect the approximations to improve as the norm of the partition P goes to zero, where the norm of P is the largest value of $\Delta\theta_k$. We are then led to the following formula defining the region's area: