### Graphing Limaçons

Graph the limaçons in Exercises 21–24. Limaçon ("*lee*-ma-sahn") is Old French for "snail." You will understand the name when you graph the limaçons in Exercise 21. Equations for limaçons have the form $r = a \pm b\cos\theta$ or $r = a \pm b\sin\theta$. There are four basic shapes.

**21. Limaçons with an inner loop**

**a.** $r = \frac{1}{2} + \cos\theta$ **b.** $r = \frac{1}{2} + \sin\theta$

**22. Cardioids**

**a.** $r = 1 - \cos\theta$ **b.** $r = -1 + \sin\theta$

**23. Dimpled limaçons**

**a.** $r = \frac{3}{2} + \cos\theta$ **b.** $r = \frac{3}{2} - \sin\theta$

**24. Oval limaçons**

**a.** $r = 2 + \cos\theta$ **b.** $r = -2 + \sin\theta$

### Graphing Polar Regions and Curves in the *xy*-Plane

**25.** Sketch the region defined by the inequalities $-1 \le r \le 2$ and $-\pi/2 \le \theta \le \pi/2$.

**26.** Sketch the region defined by the inequalities $0 \le r \le 2\sec\theta$ and $-\pi/4 \le \theta \le \pi/4$.

In Exercises 27 and 28, sketch the region defined by the inequality.

**27.** $0 \le r \le 2 - 2\cos\theta$ **28.** $0 \le r^2 \le \cos\theta$

T **29.** Which of the following has the same graph as $r = 1 - \cos\theta$?

**a.** $r = -1 - \cos\theta$ **b.** $r = 1 + \cos\theta$

Confirm your answer with algebra.

T **30.** Which of the following has the same graph as $r = \cos 2\theta$?

**a.** $r = -\sin(2\theta + \pi/2)$ **b.** $r = -\cos(\theta/2)$

Confirm your answer with algebra.

T **31. A rose within a rose** Graph the equation $r = 1 - 2\sin 3\theta$.

T **32. The nephroid of Freeth** Graph the nephroid of Freeth:

$$r = 1 + 2\sin\frac{\theta}{2}.$$

T **33. Roses** Graph the roses $r = \cos m\theta$ for $m = 1/3, 2, 3$, and 7.

T **34. Spirals** Polar coordinates are just the thing for defining spirals. Graph the following spirals.

**a.** $r = \theta$

**b.** $r = -\theta$

**c.** *A logarithmic spiral:* $r = e^{\theta/10}$

**d.** *A hyperbolic spiral:* $r = 8/\theta$

**e.** *An equilateral hyperbola:* $r = \pm 10/\sqrt{\theta}$

(Use different colors for the two branches.)

T **35.** Graph the equation $r = \sin\left(\frac{8}{7}\theta\right)$ for $0 \le \theta \le 14\pi$.

T **36.** Graph the equation

$$r = \sin^2(2.3\theta) + \cos^4(2.3\theta)$$

for $0 \le \theta \le 10\pi$.

# 11.5 Areas and Lengths in Polar Coordinates

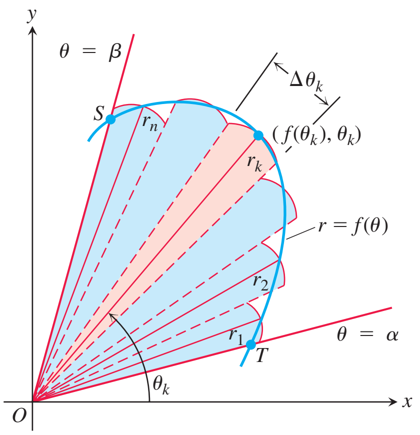



**FIGURE 11.31** To derive a formula for the area of region *OTS*, we approximate the region with fan-shaped circular sectors.

This section shows how to calculate areas of plane regions and lengths of curves in polar coordinates. The defining ideas are the same as before, but the formulas are different in polar versus Cartesian coordinates.

## Area in the Plane

The region *OTS* in Figure 11.31 is bounded by the rays $\theta = \alpha$ and $\theta = \beta$ and the curve $r = f(\theta)$. We approximate the region with $n$ nonoverlapping fan-shaped circular sectors based on a partition $P$ of angle *TOS*. The typical sector has radius $r_k = f(\theta_k)$ and central angle of radian measure $\Delta\theta_k$. Its area is $\Delta\theta_k/2\pi$ times the area of a circle of radius $r_k$, or

$$A_k = \frac{1}{2}r_k{}^2\,\Delta\theta_k = \frac{1}{2}\left(f(\theta_k)\right)^2\Delta\theta_k.$$

The area of region *OTS* is approximately

$$\sum_{k=1}^{n} A_k = \sum_{k=1}^{n}\frac{1}{2}\left(f(\theta_k)\right)^2\Delta\theta_k.$$

If $f$ is continuous, we expect the approximations to improve as the norm of the partition $P$ goes to zero, where the norm of $P$ is the largest value of $\Delta\theta_k$. We are then led to the following formula defining the region's area:

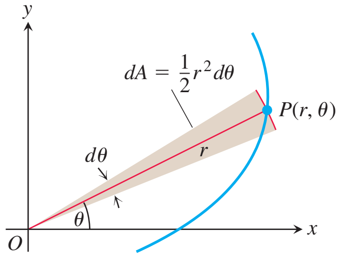



**FIGURE 11.32** The area differential $dA$ for the curve $r = f(\theta)$.

$$A = \lim_{\|P\| \to 0} \sum_{k=1}^{n} \frac{1}{2}\big(f(\theta_k)\big)^2 \, \Delta\theta_k$$

$$= \int_{\alpha}^{\beta} \frac{1}{2}\big(f(\theta)\big)^2 \, d\theta.$$

**Area of the Fan-Shaped Region Between the Origin and the Curve $r = f(\theta)$, $\alpha \le \theta \le \beta$**

$$A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta$$

This is the integral of the **area differential** (Figure 11.32)

$$dA = \frac{1}{2} r^2 \, d\theta = \frac{1}{2}\big(f(\theta)\big)^2 \, d\theta.$$

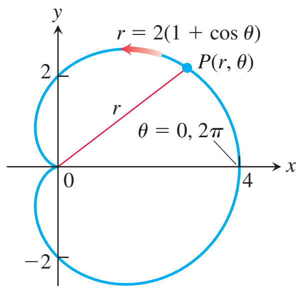



**FIGURE 11.33** The cardioid in Example 1.

**EXAMPLE 1** Find the area of the region in the $xy$-plane enclosed by the cardioid $r = 2(1 + \cos\theta)$.

**Solution** We graph the cardioid (Figure 11.33) and determine that the radius $OP$ sweeps out the region exactly once as $\theta$ runs from 0 to $2\pi$. The area is therefore

$$\int_{\theta=0}^{\theta=2\pi} \frac{1}{2} r^2 \, d\theta = \int_0^{2\pi} \frac{1}{2} \cdot 4(1 + \cos\theta)^2 \, d\theta$$

$$= \int_0^{2\pi} 2(1 + 2\cos\theta + \cos^2\theta) \, d\theta$$

$$= \int_0^{2\pi} \left(2 + 4\cos\theta + 2 \cdot \frac{1 + \cos 2\theta}{2}\right) d\theta$$

$$= \int_0^{2\pi} (3 + 4\cos\theta + \cos 2\theta) \, d\theta$$

$$= \left[3\theta + 4\sin\theta + \frac{\sin 2\theta}{2}\right]_0^{2\pi} = 6\pi - 0 = 6\pi.$$

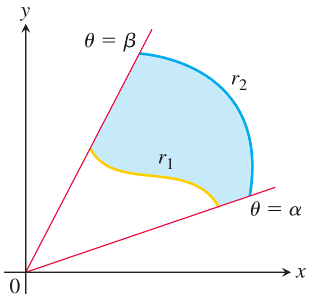



**FIGURE 11.34** The area of the shaded region is calculated by subtracting the area of the region between $r_1$ and the origin from the area of the region between $r_2$ and the origin.

To find the area of a region like the one in Figure 11.34, which lies between two polar curves $r_1 = r_1(\theta)$ and $r_2 = r_2(\theta)$ from $\theta = \alpha$ to $\theta = \beta$, we subtract the integral of $(1/2)r_1^2 \, d\theta$ from the integral of $(1/2)r_2^2 \, d\theta$. This leads to the following formula.

**Area of the Region $0 \le r_1(\theta) \le r \le r_2(\theta)$, $\alpha \le \theta \le \beta$**

$$A = \int_{\alpha}^{\beta} \frac{1}{2} r_2^2 \, d\theta - \int_{\alpha}^{\beta} \frac{1}{2} r_1^2 \, d\theta = \int_{\alpha}^{\beta} \frac{1}{2}\left(r_2^2 - r_1^2\right) d\theta \qquad (1)$$

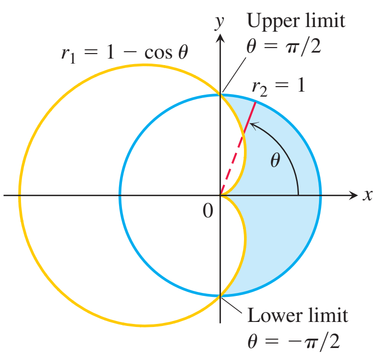



**FIGURE 11.35** The region and limits of integration in Example 2.

**EXAMPLE 2** Find the area of the region that lies inside the circle $r = 1$ and outside the cardioid $r = 1 - \cos\theta$.

**Solution** We sketch the region to determine its boundaries and find the limits of integration (Figure 11.35). The outer curve is $r_2 = 1$, the inner curve is $r_1 = 1 - \cos\theta$, and $\theta$ runs from $-\pi/2$ to $\pi/2$. The area, from Equation (1), is

$$A = \int_{-\pi/2}^{\pi/2} \frac{1}{2}\left(r_2^{\,2} - r_1^{\,2}\right) d\theta$$

$$= 2\int_{0}^{\pi/2} \frac{1}{2}\left(r_2^{\,2} - r_1^{\,2}\right) d\theta \qquad \text{Symmetry}$$

$$= \int_{0}^{\pi/2} (1 - (1 - 2\cos\theta + \cos^2\theta))\, d\theta \qquad \text{Square } r_1.$$

$$= \int_{0}^{\pi/2} (2\cos\theta - \cos^2\theta)\, d\theta = \int_{0}^{\pi/2}\left(2\cos\theta - \frac{1 + \cos 2\theta}{2}\right) d\theta$$

$$= \left[2\sin\theta - \frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^{\pi/2} = 2 - \frac{\pi}{4}.$$

The fact that we can represent a point in different ways in polar coordinates requires extra care in deciding when a point lies on the graph of a polar equation and in determining the points in which polar graphs intersect. (We needed intersection points in Example 2.) In Cartesian coordinates, we can always find the points where two curves cross by solving their equations simultaneously. In polar coordinates, the story is different. Simultaneous solution may reveal some intersection points without revealing others, so it is sometimes difficult to find all points of intersection of two polar curves. One way to identify all the points of intersection is to graph the equations.

## Length of a Polar Curve

We can obtain a polar coordinate formula for the length of a curve $r = f(\theta)$, $\alpha \le \theta \le \beta$, by parametrizing the curve as

$$x = r\cos\theta = f(\theta)\cos\theta, \qquad y = r\sin\theta = f(\theta)\sin\theta, \qquad \alpha \le \theta \le \beta. \tag{2}$$

The parametric length formula, Equation (3) from Section 11.2, then gives the length as

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta.$$

This equation becomes

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

when Equations (2) are substituted for $x$ and $y$ (Exercise 29).

**Length of a Polar Curve**

If $r = f(\theta)$ has a continuous first derivative for $\alpha \le \theta \le \beta$ and if the point $P(r, \theta)$ traces the curve $r = f(\theta)$ exactly once as $\theta$ runs from $\alpha$ to $\beta$, then the length of the curve is

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta. \tag{3}$$

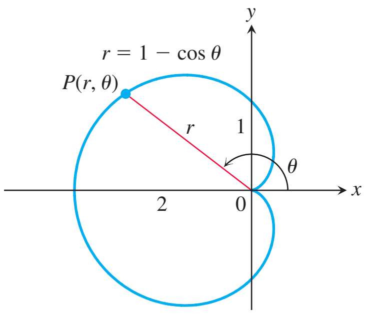



**FIGURE 11.36** Calculating the length of a cardioid (Example 3).

**EXAMPLE 3** Find the length of the cardioid $r = 1 - \cos\theta$.

**Solution** We sketch the cardioid to determine the limits of integration (Figure 11.36). The point $P(r, \theta)$ traces the curve once, counterclockwise as $\theta$ runs from 0 to $2\pi$, so these are the values we take for $\alpha$ and $\beta$.

With

$$r = 1 - \cos\theta, \qquad \frac{dr}{d\theta} = \sin\theta,$$

we have

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = (1 - \cos\theta)^2 + (\sin\theta)^2$$
$$= 1 - 2\cos\theta + \underbrace{\cos^2\theta + \sin^2\theta}_{1} = 2 - 2\cos\theta$$

and

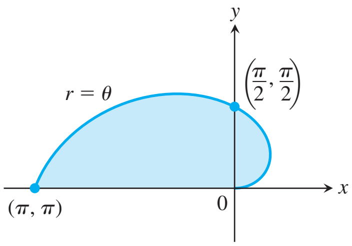
$$L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta = \int_0^{2\pi} \sqrt{2 - 2\cos\theta}\,d\theta$$
$$= \int_0^{2\pi} \sqrt{4\sin^2\frac{\theta}{2}}\,d\theta \qquad 1 - \cos\theta = 2\sin^2(\theta/2)$$
$$= \int_0^{2\pi} 2\left|\sin\frac{\theta}{2}\right|\,d\theta$$
$$= \int_0^{2\pi} 2\sin\frac{\theta}{2}\,d\theta \qquad \sin(\theta/2) \ge 0 \quad \text{for} \quad 0 \le \theta \le 2\pi$$
$$= \left[-4\cos\frac{\theta}{2}\right]_0^{2\pi} = 4 + 4 = 8.$$

## Exercises 11.5

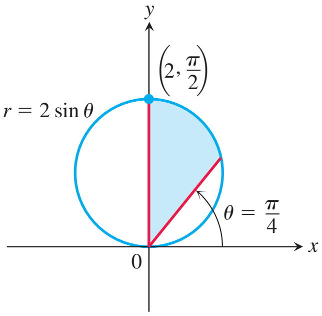
### Finding Polar Areas

Find the areas of the regions in Exercises 1–8.

1. Bounded by the spiral $r = \theta$ for $0 \le \theta \le \pi$




2. Bounded by the circle $r = 2\sin\theta$ for $\pi/4 \le \theta \le \pi/2$

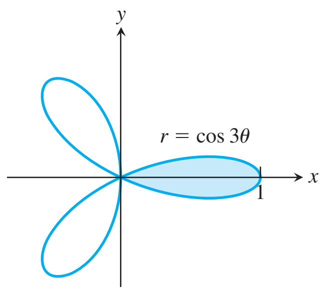



3. Inside the oval limaçon $r = 4 + 2\cos\theta$

4. Inside the cardioid $r = a(1 + \cos\theta)$, $a > 0$

5. Inside one leaf of the four-leaved rose $r = \cos 2\theta$

6. Inside one leaf of the three-leaved rose $r = \cos 3\theta$




7. Inside one loop of the lemniscate $r^2 = 4\sin 2\theta$

8. Inside the six-leaved rose $r^2 = 2\sin 3\theta$

Find the areas of the regions in Exercises 9–18.

9. Shared by the circles $r = 2\cos\theta$ and $r = 2\sin\theta$

10. Shared by the circles $r = 1$ and $r = 2\sin\theta$

11. Shared by the circle $r = 2$ and the cardioid $r = 2(1 - \cos\theta)$

12. Shared by the cardioids $r = 2(1 + \cos\theta)$ and $r = 2(1 - \cos\theta)$

13. Inside the lemniscate $r^2 = 6\cos 2\theta$ and outside the circle $r = \sqrt{3}$

**14.** Inside the circle $r = 3a\cos\theta$ and outside the cardioid $r = a(1 + \cos\theta)$, $a > 0$

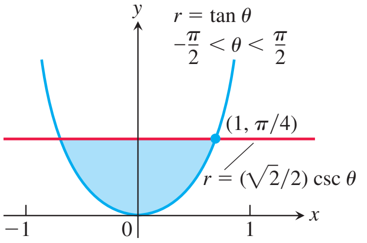
**15.** Inside the circle $r = -2\cos\theta$ and outside the circle $r = 1$

**16.** Inside the circle $r = 6$ above the line $r = 3\csc\theta$

**17.** Inside the circle $r = 4\cos\theta$ and to the right of the vertical line $r = \sec\theta$

**18.** Inside the circle $r = 4\sin\theta$ and below the horizontal line $r = 3\csc\theta$

**19. a.** Find the area of the shaded region in the accompanying figure.




**b.** It looks as if the graph of $r = \tan\theta, -\pi/2 < \theta < \pi/2$, could be asymptotic to the lines $x = 1$ and $x = -1$. Is it? Give reasons for your answer.

**20.** The area of the region that lies inside the cardioid curve $r = \cos\theta + 1$ and outside the circle $r = \cos\theta$ is not

$$\frac{1}{2}\int_0^{2\pi}\left[(\cos\theta + 1)^2 - \cos^2\theta\right]d\theta = \pi.$$

Why not? What *is* the area? Give reasons for your answers.

### Finding Lengths of Polar Curves

Find the lengths of the curves in Exercises 21–28.

**21.** The spiral $r = \theta^2$, $0 \le \theta \le \sqrt{5}$

**22.** The spiral $r = e^{\theta}/\sqrt{2}$, $0 \le \theta \le \pi$

**23.** The cardioid $r = 1 + \cos\theta$

**24.** The curve $r = a\sin^2(\theta/2)$, $0 \le \theta \le \pi$, $a > 0$

**25.** The parabolic segment $r = 6/(1 + \cos\theta)$, $0 \le \theta \le \pi/2$

**26.** The parabolic segment $r = 2/(1 - \cos\theta)$, $\pi/2 \le \theta \le \pi$

**27.** The curve $r = \cos^3(\theta/3)$, $0 \le \theta \le \pi/4$

**28.** The curve $r = \sqrt{1 + \sin 2\theta}$, $0 \le \theta \le \pi\sqrt{2}$

**29. The length of the curve $r = f(\theta)$, $\alpha \le \theta \le \beta$** Assuming that the necessary derivatives are continuous, show how the substitutions

$$x = f(\theta)\cos\theta, \quad y = f(\theta)\sin\theta$$

(Equations 2 in the text) transform

$$L = \int_{\alpha}^{\beta}\sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\,d\theta$$

into

$$L = \int_{\alpha}^{\beta}\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta.$$

**30. Circumferences of circles** As usual, when faced with a new formula, it is a good idea to try it on familiar objects to be sure it gives results consistent with past experience. Use the length formula in Equation (3) to calculate the circumferences of the following circles ($a > 0$).

**a.** $r = a$ **b.** $r = a\cos\theta$ **c.** $r = a\sin\theta$

### Theory and Examples

**31. Average value** If $f$ is continuous, the average value of the polar coordinate $r$ over the curve $r = f(\theta)$, $\alpha \le \theta \le \beta$, with respect to $\theta$ is given by the formula

$$r_{\text{av}} = \frac{1}{\beta - \alpha}\int_{\alpha}^{\beta} f(\theta)\,d\theta.$$

Use this formula to find the average value of $r$ with respect to $\theta$ over the following curves ($a > 0$).

**a.** The cardioid $r = a(1 - \cos\theta)$

**b.** The circle $r = a$

**c.** The circle $r = a\cos\theta$, $-\pi/2 \le \theta \le \pi/2$

**32. $r = f(\theta)$ *vs.* $r = 2f(\theta)$** Can anything be said about the relative lengths of the curves $r = f(\theta)$, $\alpha \le \theta \le \beta$, and $r = 2f(\theta)$, $\alpha \le \theta \le \beta$? Give reasons for your answer.

# 11.6 Conic Sections

In this section we define and review parabolas, ellipses, and hyperbolas geometrically and derive their standard Cartesian equations. These curves are called *conic sections* or *conics* because they are formed by cutting a double cone with a plane (Figure 11.37). This geometry method was the only way they could be described by Greek mathematicians who did not have our tools of Cartesian or polar coordinates. In the next section we express the conics in polar coordinates.

## Parabolas

**DEFINITIONS** A set that consists of all the points in a plane equidistant from a given fixed point and a given fixed line in the plane is a **parabola**. The fixed point is the **focus** of the parabola. The fixed line is the **directrix**.

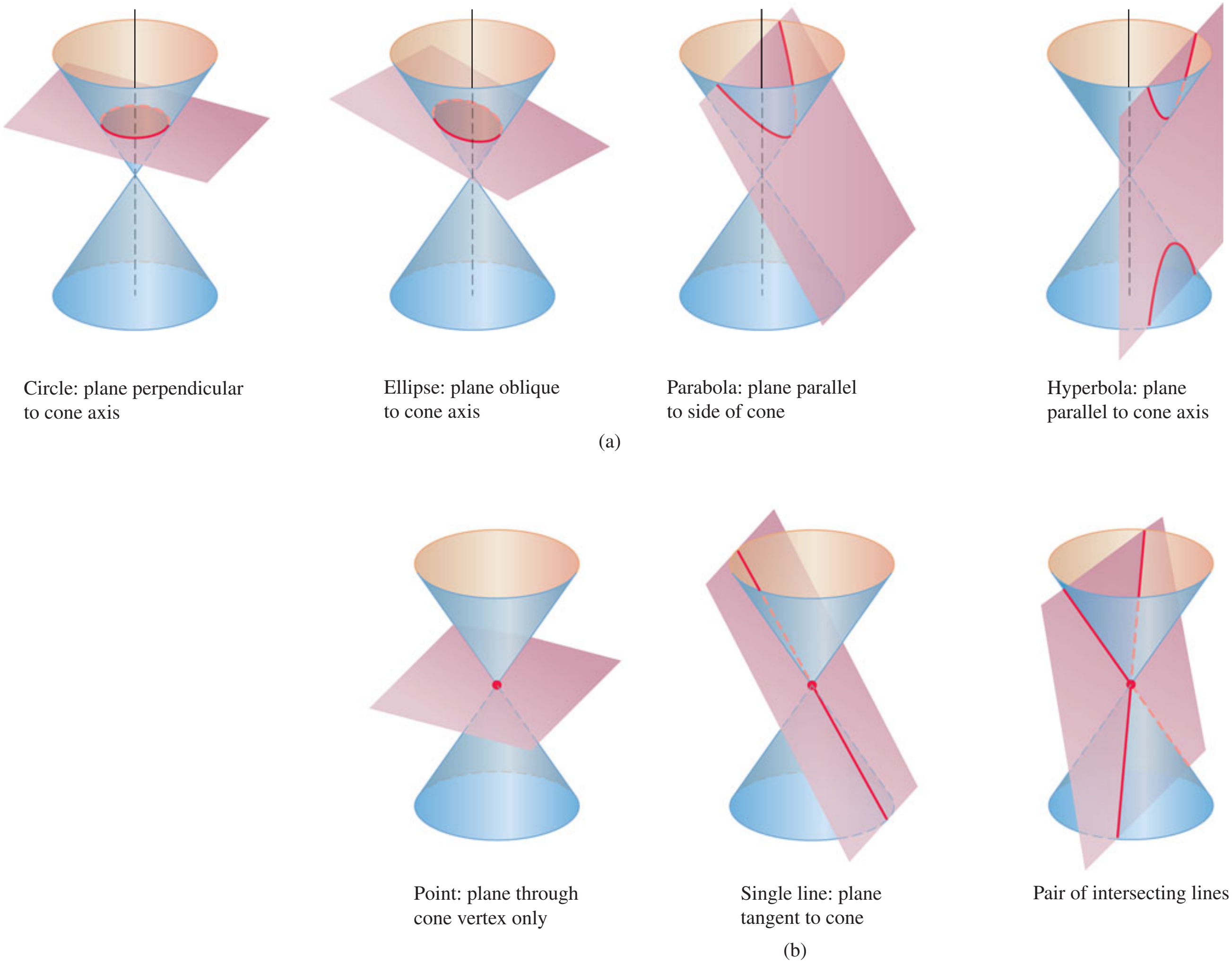



**FIGURE 11.37** The standard conic sections (a) are the curves in which a plane cuts a *double* cone. Hyperbolas come in two parts, called *branches*. The point and lines obtained by passing the plane through the cone's vertex (b) are *degenerate* conic sections.

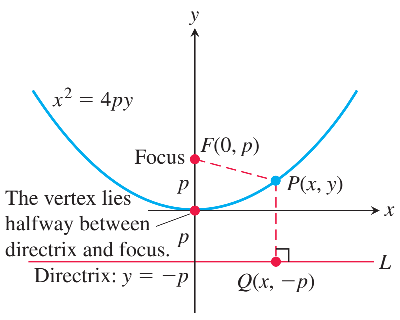



**FIGURE 11.38** The standard form of the parabola $x^2 = 4py$, $p > 0$.

If the focus $F$ lies on the directrix $L$, the parabola is the line through $F$ perpendicular to $L$. We consider this to be a degenerate case and assume henceforth that $F$ does not lie on $L$.

A parabola has its simplest equation when its focus and directrix straddle one of the coordinate axes. For example, suppose that the focus lies at the point $F(0, p)$ on the positive $y$-axis and that the directrix is the line $y = -p$ (Figure 11.38). In the notation of the figure, a point $P(x, y)$ lies on the parabola if and only if $PF = PQ$. From the distance formula,

$$PF = \sqrt{(x - 0)^2 + (y - p)^2} = \sqrt{x^2 + (y - p)^2}$$
$$PQ = \sqrt{(x - x)^2 + (y - (-p))^2} = \sqrt{(y + p)^2}.$$

When we equate these expressions, square, and simplify, we get

$$y = \frac{x^2}{4p} \quad \text{or} \quad x^2 = 4py. \qquad \text{Standard form} \qquad (1)$$

These equations reveal the parabola's symmetry about the $y$-axis. We call the $y$-axis the **axis** of the parabola (short for "axis of symmetry").

The point where a parabola crosses its axis is the **vertex**. The vertex of the parabola $x^2 = 4py$ lies at the origin (Figure 11.38). The positive number $p$ is the parabola's **focal length**.

If the parabola opens downward, with its focus at $(0, -p)$ and its directrix the line $y = p$, then Equations (1) become

$$y = -\frac{x^2}{4p} \qquad \text{and} \qquad x^2 = -4py.$$

By interchanging the variables $x$ and $y$, we obtain similar equations for parabolas opening to the right or to the left (Figure 11.39).

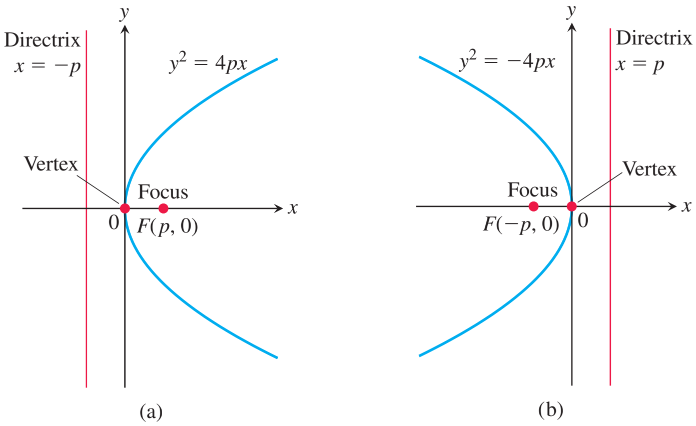



**FIGURE 11.39** (a) The parabola $y^2 = 4px$. (b) The parabola $y^2 = -4px$.

**EXAMPLE 1** Find the focus and directrix of the parabola $y^2 = 10x$.

**Solution** We find the value of $p$ in the standard equation $y^2 = 4px$:

$$4p = 10, \qquad \text{so} \qquad p = \frac{10}{4} = \frac{5}{2}.$$

Then we find the focus and directrix for this value of $p$:

$$\text{Focus:} \quad (p, 0) = \left(\frac{5}{2}, 0\right)$$

$$\text{Directrix:} \quad x = -p \qquad \text{or} \qquad x = -\frac{5}{2}.$$

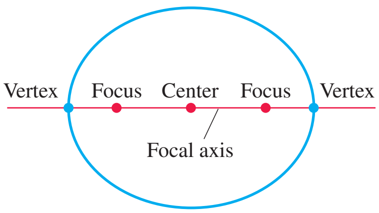



**FIGURE 11.40** Points on the focal axis of an ellipse.

## Ellipses

**DEFINITIONS** An **ellipse** is the set of points in a plane whose distances from two fixed points in the plane have a constant sum. The two fixed points are the **foci** of the ellipse.

The line through the foci of an ellipse is the ellipse's **focal axis**. The point on the axis halfway between the foci is the **center**. The points where the focal axis and ellipse cross are the ellipse's **vertices** (Figure 11.40).

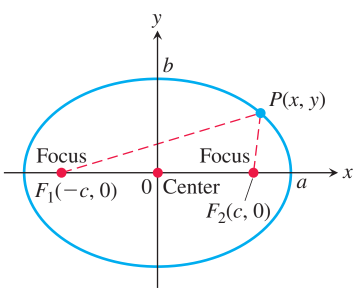



**FIGURE 11.41** The ellipse defined by the equation $PF_1 + PF_2 = 2a$ is the graph of the equation $(x^2/a^2) + (y^2/b^2) = 1$, where $b^2 = a^2 - c^2$.

If the foci are $F_1(-c, 0)$ and $F_2(c, 0)$ (Figure 11.41), and $PF_1 + PF_2$ is denoted by $2a$, then the coordinates of a point $P$ on the ellipse satisfy the equation

$$\sqrt{(x + c)^2 + y^2} + \sqrt{(x - c)^2 + y^2} = 2a.$$

To simplify this equation, we move the second radical to the right-hand side, square, isolate the remaining radical, and square again, obtaining

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1. \tag{2}$$

Since $PF_1 + PF_2$ is greater than the length $F_1F_2$ (by the triangle inequality for triangle $PF_1F_2$), the number $2a$ is greater than $2c$. Accordingly, $a > c$ and the number $a^2 - c^2$ in Equation (2) is positive.

The algebraic steps leading to Equation (2) can be reversed to show that every point $P$ whose coordinates satisfy an equation of this form with $0 < c < a$ also satisfies the equation $PF_1 + PF_2 = 2a$. A point therefore lies on the ellipse if and only if its coordinates satisfy Equation (2).

If

$$b = \sqrt{a^2 - c^2}, \tag{3}$$

then $a^2 - c^2 = b^2$ and Equation (2) takes the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \tag{4}$$

Equation (4) reveals that this ellipse is symmetric with respect to the origin and both coordinate axes. It lies inside the rectangle bounded by the lines $x = \pm a$ and $y = \pm b$. It crosses the axes at the points $(\pm a, 0)$ and $(0, \pm b)$. The tangents at these points are perpendicular to the axes because

$$\frac{dy}{dx} = -\frac{b^2x}{a^2y}$$

Obtained from Eq. (4) by implicit differentiation

is zero if $x = 0$ and infinite if $y = 0$.

The **major axis** of the ellipse in Equation (4) is the line segment of length $2a$ joining the points $(\pm a, 0)$. The **minor axis** is the line segment of length $2b$ joining the points $(0, \pm b)$. The number $a$ itself is the **semimajor axis**, the number $b$ the **semiminor axis**. The number $c$, found from Equation (3) as

$$c = \sqrt{a^2 - b^2},$$

is the **center-to-focus distance** of the ellipse. If $a = b$, the ellipse is a circle.

**EXAMPLE 2** The ellipse

$$\frac{x^2}{16} + \frac{y^2}{9} = 1 \tag{5}$$

(Figure 11.42) has

Semimajor axis: $a = \sqrt{16} = 4$, Semiminor axis: $b = \sqrt{9} = 3$

Center-to-focus distance: $c = \sqrt{16 - 9} = \sqrt{7}$

Foci: $(\pm c, 0) = \left(\pm\sqrt{7}, 0\right)$

Vertices: $(\pm a, 0) = (\pm 4, 0)$

Center: $(0, 0)$. ■

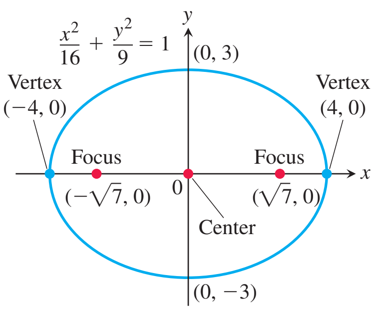



**FIGURE 11.42** An ellipse with its major axis horizontal (Example 2).

If we interchange $x$ and $y$ in Equation (5), we have the equation

$$\frac{x^2}{9} + \frac{y^2}{16} = 1. \tag{6}$$

The major axis of this ellipse is now vertical instead of horizontal, with the foci and vertices on the $y$-axis. There is no confusion in analyzing Equations (5) and (6). If we find the intercepts on the coordinate axes, we will know which way the major axis runs because it is the longer of the two axes.

**Standard-Form Equations for Ellipses Centered at the Origin**

*Foci on the x-axis:* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (a > b)$

Center-to-focus distance: $c = \sqrt{a^2 - b^2}$

Foci: $(\pm c, 0)$

Vertices: $(\pm a, 0)$

*Foci on the y-axis:* $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 \quad (a > b)$

Center-to-focus distance: $c = \sqrt{a^2 - b^2}$

Foci: $(0, \pm c)$

Vertices: $(0, \pm a)$

In each case, $a$ is the semimajor axis and $b$ is the semiminor axis.

## Hyperbolas

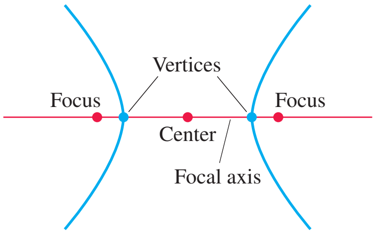



**FIGURE 11.43** Points on the focal axis of a hyperbola.

**DEFINITIONS** A **hyperbola** is the set of points in a plane whose distances from two fixed points in the plane have a constant difference. The two fixed points are the **foci** of the hyperbola.

The line through the foci of a hyperbola is the **focal axis**. The point on the axis halfway between the foci is the hyperbola's **center**. The points where the focal axis and hyperbola cross are the **vertices** (Figure 11.43).

If the foci are $F_1(-c, 0)$ and $F_2(c, 0)$ (Figure 11.44) and the constant difference is $2a$, then a point $(x, y)$ lies on the hyperbola if and only if

$$\sqrt{(x + c)^2 + y^2} - \sqrt{(x - c)^2 + y^2} = \pm 2a. \tag{7}$$

To simplify this equation, we move the second radical to the right-hand side, square, isolate the remaining radical, and square again, obtaining

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1. \tag{8}$$

So far, this looks just like the equation for an ellipse. But now $a^2 - c^2$ is negative because $2a$, being the difference of two sides of triangle $PF_1F_2$, is less than $2c$, the third side.

The algebraic steps leading to Equation (8) can be reversed to show that every point $P$ whose coordinates satisfy an equation of this form with $0 < a < c$ also satisfies Equation (7). A point therefore lies on the hyperbola if and only if its coordinates satisfy Equation (8).

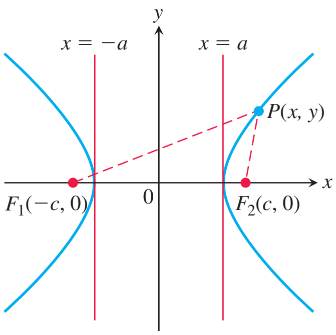



**FIGURE 11.44** Hyperbolas have two branches. For points on the right-hand branch of the hyperbola shown here, $PF_1 - PF_2 = 2a$. For points on the left-hand branch, $PF_2 - PF_1 = 2a$. We then let $b = \sqrt{c^2 - a^2}$.

If we let $b$ denote the positive square root of $c^2 - a^2$,

$$b = \sqrt{c^2 - a^2}, \tag{9}$$

then $a^2 - c^2 = -b^2$ and Equation (8) takes the more compact form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1. \tag{10}$$

The differences between Equation (10) and the equation for an ellipse (Equation 4) are the minus sign and the new relation

$$c^2 = a^2 + b^2. \qquad \text{From Eq. (9)}$$

Like the ellipse, the hyperbola is symmetric with respect to the origin and coordinate axes. It crosses the $x$-axis at the points $(\pm a, 0)$. The tangents at these points are vertical because

$$\frac{dy}{dx} = \frac{b^2x}{a^2y} \qquad \text{Obtained from Eq. (10) by implicit differentiation}$$

is infinite when $y = 0$. The hyperbola has no $y$-intercepts; in fact, no part of the curve lies between the lines $x = -a$ and $x = a$.

The lines

$$y = \pm\frac{b}{a}x$$

are the two **asymptotes** of the hyperbola defined by Equation (10). The fastest way to find the equations of the asymptotes is to replace the 1 in Equation (10) by 0 and solve the new equation for $y$:

$$\underbrace{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}_{\text{hyperbola}} \rightarrow \underbrace{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0}_{\text{0 for 1}} \rightarrow \underbrace{y = \pm\frac{b}{a}x.}_{\text{asymptotes}}$$

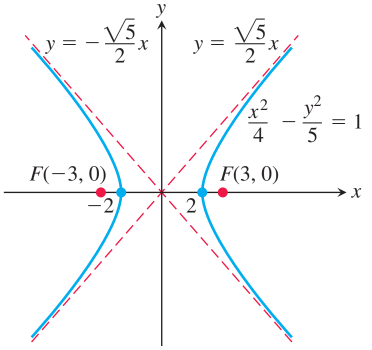



**FIGURE 11.45** The hyperbola and its asymptotes in Example 3.

**EXAMPLE 3** The equation

$$\frac{x^2}{4} - \frac{y^2}{5} = 1 \tag{11}$$

is Equation (10) with $a^2 = 4$ and $b^2 = 5$ (Figure 11.45). We have

Center-to-focus distance: $c = \sqrt{a^2 + b^2} = \sqrt{4 + 5} = 3$

Foci: $(\pm c, 0) = (\pm 3, 0)$, Vertices: $(\pm a, 0) = (\pm 2, 0)$

Center: $(0, 0)$

Asymptotes: $\frac{x^2}{4} - \frac{y^2}{5} = 0$ or $y = \pm\frac{\sqrt{5}}{2}x.$ ■

If we interchange $x$ and $y$ in Equation (11), the foci and vertices of the resulting hyperbola will lie along the $y$-axis. We still find the asymptotes in the same way as before, but now their equations will be $y = \pm 2x/\sqrt{5}$.

### Standard-Form Equations for Hyperbolas Centered at the Origin

*Foci on the x-axis:* $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Center-to-focus distance: $c = \sqrt{a^2 + b^2}$

Foci: $(\pm c, 0)$

Vertices: $(\pm a, 0)$

Asymptotes: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$ or $y = \pm\frac{b}{a}x$

*Foci on the y-axis:* $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$

Center-to-focus distance: $c = \sqrt{a^2 + b^2}$

Foci: $(0, \pm c)$

Vertices: $(0, \pm a)$

Asymptotes: $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 0$ or $y = \pm\frac{a}{b}x$

Notice the difference in the asymptote equations ($b/a$ in the first, $a/b$ in the second).

We shift conics using the principles reviewed in Section 1.2, replacing $x$ by $x + h$ and $y$ by $y + k$.

**EXAMPLE 4** Show that the equation $x^2 - 4y^2 + 2x + 8y - 7 = 0$ represents a hyperbola. Find its center, asymptotes, and foci.

**Solution** We reduce the equation to standard form by completing the square in $x$ and $y$ as follows:

$$(x^2 + 2x) - 4(y^2 - 2y) = 7$$

$$(x^2 + 2x + 1) - 4(y^2 - 2y + 1) = 7 + 1 - 4$$

$$\frac{(x+1)^2}{4} - (y-1)^2 = 1.$$

This is the standard form Equation (10) of a hyperbola with $x$ replaced by $x + 1$ and $y$ replaced by $y - 1$. The hyperbola is shifted one unit to the left and one unit upward, and it has center $x + 1 = 0$ and $y - 1 = 0$, or $x = -1$ and $y = 1$. Moreover,

$$a^2 = 4, \qquad b^2 = 1, \qquad c^2 = a^2 + b^2 = 5,$$

so the asymptotes are the two lines

$$\frac{x+1}{2} - (y-1) = 0 \qquad \text{and} \qquad \frac{x+1}{2} + (y-1) = 0,$$

or

$$y - 1 = \pm\frac{1}{2}(x+1).$$

The shifted foci have coordinates $\left(-1 \pm \sqrt{5}, 1\right)$. ■

## Exercises 11.6

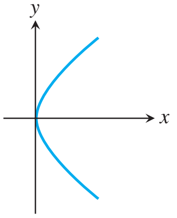
### Identifying Graphs

Match the parabolas in Exercises 1–4 with the following equations:

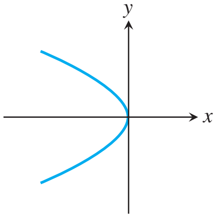
$$x^2 = 2y, \quad x^2 = -6y, \quad y^2 = 8x, \quad y^2 = -4x.$$

Then find each parabola's focus and directrix.

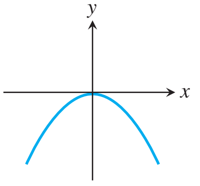
**1.**



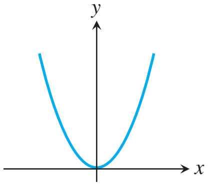

**2.**




**3.**




**4.**



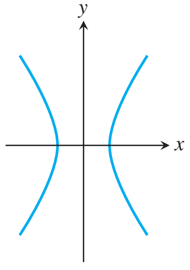

Match each conic section in Exercises 5–8 with one of these equations:

$$\frac{x^2}{4} + \frac{y^2}{9} = 1, \qquad \frac{x^2}{2} + y^2 = 1,$$

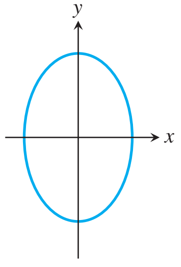
$$\frac{y^2}{4} - x^2 = 1, \qquad \frac{x^2}{4} - \frac{y^2}{9} = 1.$$

Then find the conic section's foci and vertices. If the conic section is a hyperbola, find its asymptotes as well.

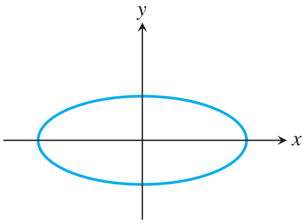
**5.**



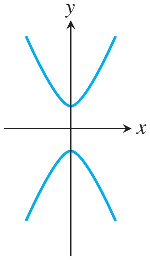

**6.**



**7.**




**8.**




## Parabolas

Exercises 9–16 give equations of parabolas. Find each parabola's focus and directrix. Then sketch the parabola. Include the focus and directrix in your sketch.

**9.** $y^2 = 12x$ **10.** $x^2 = 6y$ **11.** $x^2 = -8y$

**12.** $y^2 = -2x$ **13.** $y = 4x^2$ **14.** $y = -8x^2$

**15.** $x = -3y^2$ **16.** $x = 2y^2$

## Ellipses

Exercises 17–24 give equations for ellipses. Put each equation in standard form. Then sketch the ellipse. Include the foci in your sketch.

**17.** $16x^2 + 25y^2 = 400$ **18.** $7x^2 + 16y^2 = 112$

**19.** $2x^2 + y^2 = 2$ **20.** $2x^2 + y^2 = 4$

**21.** $3x^2 + 2y^2 = 6$ **22.** $9x^2 + 10y^2 = 90$

**23.** $6x^2 + 9y^2 = 54$ **24.** $169x^2 + 25y^2 = 4225$

Exercises 25 and 26 give information about the foci and vertices of ellipses centered at the origin of the $xy$-plane. In each case, find the ellipse's standard-form equation from the given information.

**25.** Foci: $\left(\pm\sqrt{2}, 0\right)$ Vertices: $(\pm 2, 0)$

**26.** Foci: $(0, \pm 4)$ Vertices: $(0, \pm 5)$

## Hyperbolas

Exercises 27–34 give equations for hyperbolas. Put each equation in standard form and find the hyperbola's asymptotes. Then sketch the hyperbola. Include the asymptotes and foci in your sketch.

**27.** $x^2 - y^2 = 1$ **28.** $9x^2 - 16y^2 = 144$

**29.** $y^2 - x^2 = 8$ **30.** $y^2 - x^2 = 4$

**31.** $8x^2 - 2y^2 = 16$ **32.** $y^2 - 3x^2 = 3$

**33.** $8y^2 - 2x^2 = 16$ **34.** $64x^2 - 36y^2 = 2304$

Exercises 35–38 give information about the foci, vertices, and asymptotes of hyperbolas centered at the origin of the $xy$-plane. In each case, find the hyperbola's standard-form equation from the information given.

**35.** Foci: $\left(0, \pm\sqrt{2}\right)$
Asymptotes: $y = \pm x$

**36.** Foci: $(\pm 2, 0)$
Asymptotes: $y = \pm\dfrac{1}{\sqrt{3}}x$

**37.** Vertices: $(\pm 3, 0)$
Asymptotes: $y = \pm\dfrac{4}{3}x$

**38.** Vertices: $(0, \pm 2)$
Asymptotes: $y = \pm\dfrac{1}{2}x$

## Shifting Conic Sections

You may wish to review Section 1.2 before solving Exercises 39–56.

**39.** The parabola $y^2 = 8x$ is shifted down 2 units and right 1 unit to generate the parabola $(y + 2)^2 = 8(x - 1)$.

**a.** Find the new parabola's vertex, focus, and directrix.

**b.** Plot the new vertex, focus, and directrix, and sketch in the parabola.

**40.** The parabola $x^2 = -4y$ is shifted left 1 unit and up 3 units to generate the parabola $(x + 1)^2 = -4(y - 3)$.

**a.** Find the new parabola's vertex, focus, and directrix.

**b.** Plot the new vertex, focus, and directrix, and sketch in the parabola.

**41.** The ellipse $(x^2/16) + (y^2/9) = 1$ is shifted 4 units to the right and 3 units up to generate the ellipse

$$\frac{(x-4)^2}{16} + \frac{(y-3)^2}{9} = 1.$$

**a.** Find the foci, vertices, and center of the new ellipse.

**b.** Plot the new foci, vertices, and center, and sketch in the new ellipse.

**42.** The ellipse $(x^2/9) + (y^2/25) = 1$ is shifted 3 units to the left and 2 units down to generate the ellipse

$$\frac{(x+3)^2}{9} + \frac{(y+2)^2}{25} = 1.$$

**a.** Find the foci, vertices, and center of the new ellipse.

**b.** Plot the new foci, vertices, and center, and sketch in the new ellipse.

**43.** The hyperbola $(x^2/16) - (y^2/9) = 1$ is shifted 2 units to the right to generate the hyperbola

$$\frac{(x-2)^2}{16} - \frac{y^2}{9} = 1.$$

**a.** Find the center, foci, vertices, and asymptotes of the new hyperbola.

**b.** Plot the new center, foci, vertices, and asymptotes, and sketch in the hyperbola.

**44.** The hyperbola $(y^2/4) - (x^2/5) = 1$ is shifted 2 units down to generate the hyperbola

$$\frac{(y+2)^2}{4} - \frac{x^2}{5} = 1.$$

**a.** Find the center, foci, vertices, and asymptotes of the new hyperbola.

**b.** Plot the new center, foci, vertices, and asymptotes, and sketch in the hyperbola.

Exercises 45–48 give equations for parabolas and tell how many units up or down and to the right or left each parabola is to be shifted. Find an equation for the new parabola, and find the new vertex, focus, and directrix.

**45.** $y^2 = 4x$, left 2, down 3 **46.** $y^2 = -12x$, right 4, up 3

**47.** $x^2 = 8y$, right 1, down 7 **48.** $x^2 = 6y$, left 3, down 2

Exercises 49–52 give equations for ellipses and tell how many units up or down and to the right or left each ellipse is to be shifted. Find an equation for the new ellipse, and find the new foci, vertices, and center.

**49.** $\frac{x^2}{6} + \frac{y^2}{9} = 1$, left 2, down 1

**50.** $\frac{x^2}{2} + y^2 = 1$, right 3, up 4

**51.** $\frac{x^2}{3} + \frac{y^2}{2} = 1$, right 2, up 3

**52.** $\frac{x^2}{16} + \frac{y^2}{25} = 1$, left 4, down 5

Exercises 53–56 give equations for hyperbolas and tell how many units up or down and to the right or left each hyperbola is to be shifted. Find an equation for the new hyperbola, and find the new center, foci, vertices, and asymptotes.

**53.** $\frac{x^2}{4} - \frac{y^2}{5} = 1$, right 2, up 2

**54.** $\frac{x^2}{16} - \frac{y^2}{9} = 1$, left 2, down 1

**55.** $y^2 - x^2 = 1$, left 1, down 1

**56.** $\frac{y^2}{3} - x^2 = 1$, right 1, up 3

Find the center, foci, vertices, asymptotes, and radius, as appropriate, of the conic sections in Exercises 57–68.

**57.** $x^2 + 4x + y^2 = 12$

**58.** $2x^2 + 2y^2 - 28x + 12y + 114 = 0$

**59.** $x^2 + 2x + 4y - 3 = 0$

**60.** $y^2 - 4y - 8x - 12 = 0$

**61.** $x^2 + 5y^2 + 4x = 1$

**62.** $9x^2 + 6y^2 + 36y = 0$

**63.** $x^2 + 2y^2 - 2x - 4y = -1$

**64.** $4x^2 + y^2 + 8x - 2y = -1$

**65.** $x^2 - y^2 - 2x + 4y = 4$

**66.** $x^2 - y^2 + 4x - 6y = 6$

**67.** $2x^2 - y^2 + 6y = 3$

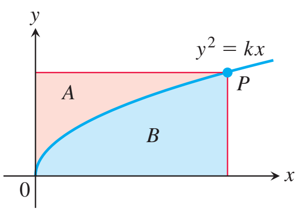
**68.** $y^2 - 4x^2 + 16x = 24$

## Theory and Examples

**69.** If lines are drawn parallel to the coordinate axes through a point $P$ on the parabola $y^2 = kx$, $k > 0$, the parabola partitions the rectangular region bounded by these lines and the coordinate axes into two smaller regions, $A$ and $B$.

**a.** If the two smaller regions are revolved about the $y$-axis, show that they generate solids whose volumes have the ratio 4:1.

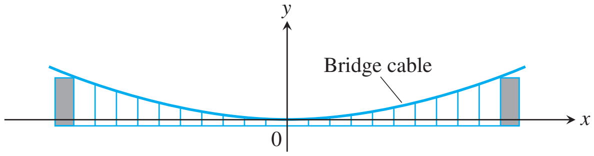
**b.** What is the ratio of the volumes generated by revolving the regions about the $x$-axis?




**70. Suspension bridge cables hang in parabolas** The suspension bridge cable shown in the accompanying figure supports a uniform load of $w$ pounds per horizontal foot. It can be shown that if $H$ is the horizontal tension of the cable at the origin, then the curve of the cable satisfies the equation

$$\frac{dy}{dx} = \frac{w}{H}x.$$

Show that the cable hangs in a parabola by solving this differential equation subject to the initial condition that $y = 0$ when $x = 0$.




**71. The width of a parabola at the focus** Show that the number $4p$ is the *width* of the parabola $x^2 = 4py$ $(p > 0)$ at the focus by showing that the line $y = p$ cuts the parabola at points that are $4p$ units apart.

**72. The asymptotes of $(x^2/a^2) - (y^2/b^2) = 1$** Show that the vertical distance between the line $y = (b/a)x$ and the upper half of the right-hand branch $y = (b/a)\sqrt{x^2 - a^2}$ of the hyperbola $(x^2/a^2) - (y^2/b^2) = 1$ approaches 0 by showing that

$$\lim_{x\to\infty}\left(\frac{b}{a}x - \frac{b}{a}\sqrt{x^2 - a^2}\right) = \frac{b}{a}\lim_{x\to\infty}\left(x - \sqrt{x^2 - a^2}\right) = 0.$$

Similar results hold for the remaining portions of the hyperbola and the lines $y = \pm(b/a)x$.

**73. Area** Find the dimensions of the rectangle of largest area that can be inscribed in the ellipse $x^2 + 4y^2 = 4$ with its sides parallel to the coordinate axes. What is the area of the rectangle?

**74. Volume** Find the volume of the solid generated by revolving the region enclosed by the ellipse $9x^2 + 4y^2 = 36$ about the **(a)** $x$-axis, **(b)** $y$-axis.

**75. Volume** The "triangular" region in the first quadrant bounded by the $x$-axis, the line $x = 4$, and the hyperbola $9x^2 - 4y^2 = 36$ is revolved about the $x$-axis to generate a solid. Find the volume of the solid.

**76. Tangents** Show that the tangents to the curve $y^2 = 4px$ from any point on the line $x = -p$ are perpendicular.

**77. Tangents** Find equations for the tangents to the circle $(x - 2)^2 + (y - 1)^2 = 5$ at the points where the circle crosses the coordinate axes.

**78. Volume** The region bounded on the left by the $y$-axis, on the right by the hyperbola $x^2 - y^2 = 1$, and above and below by the lines $y = \pm 3$ is revolved about the $y$-axis to generate a solid. Find the volume of the solid.

**79. Centroid** Find the centroid of the region that is bounded below by the $x$-axis and above by the ellipse $(x^2/9) + (y^2/16) = 1$.

**80. Surface area** The curve $y = \sqrt{x^2 + 1}$, $0 \le x \le \sqrt{2}$, which is part of the upper branch of the hyperbola $y^2 - x^2 = 1$, is revolved about the $x$-axis to generate a surface. Find the area of the surface.

**81. The reflective property of parabolas** The accompanying figure shows a typical point $P(x_0, y_0)$ on the parabola $y^2 = 4px$. The line $L$ is tangent to the parabola at $P$. The parabola's focus lies at $F(p, 0)$. The ray $L'$ extending from $P$ to the right is parallel to the $x$-axis. We show that light from $F$ to $P$ will be reflected out along $L'$ by showing that $\beta$ equals $\alpha$. Establish this equality by taking the following steps.

**a.** Show that $\tan\beta = 2p/y_0$.

**b.** Show that $\tan\phi = y_0/(x_0 - p)$.

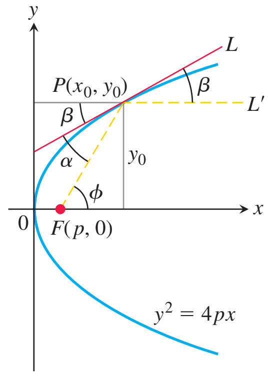
**c.** Use the identity

$$\tan\alpha = \frac{\tan\phi - \tan\beta}{1 + \tan\phi\tan\beta}$$

to show that $\tan\alpha = 2p/y_0$.

Since $\alpha$ and $\beta$ are both acute, $\tan\beta = \tan\alpha$ implies $\beta = \alpha$.

This reflective property of parabolas is used in applications like car headlights, radio telescopes, and satellite TV dishes.




# 11.7 Conics in Polar Coordinates

Polar coordinates are especially important in astronomy and astronautical engineering because satellites, moons, planets, and comets all move approximately along ellipses, parabolas, and hyperbolas that can be described with a single relatively simple polar coordinate equation. We develop that equation here after first introducing the idea of a conic section's *eccentricity*. The eccentricity reveals the conic section's type (circle, ellipse, parabola, or hyperbola) and the degree to which it is "squashed" or flattened.

## Eccentricity

Although the center-to-focus distance $c$ does not appear in the standard Cartesian equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \qquad (a > b)$$

for an ellipse, we can still determine $c$ from the equation $c = \sqrt{a^2 - b^2}$. If we fix $a$ and vary $c$ over the interval $0 \le c \le a$, the resulting ellipses will vary in shape. They are circles if $c = 0$ (so that $a = b$) and flatten, becoming more oblong, as $c$ increases. If $c = a$, the foci and vertices overlap and the ellipse degenerates into a line segment. Thus we are led to consider the ratio $e = c/a$. We use this ratio for hyperbolas as well, except in this case $c$ equals $\sqrt{a^2 + b^2}$ instead of $\sqrt{a^2 - b^2}$. We define these ratios with the term *eccentricity*.

**DEFINITION**

The **eccentricity** of the ellipse $(x^2/a^2) + (y^2/b^2) = 1$ $(a > b)$ is

$$e = \frac{c}{a} = \frac{\sqrt{a^2 - b^2}}{a}.$$

The **eccentricity** of the hyperbola $(x^2/a^2) - (y^2/b^2) = 1$ is

$$e = \frac{c}{a} = \frac{\sqrt{a^2 + b^2}}{a}.$$

The **eccentricity** of a parabola is $e = 1$.

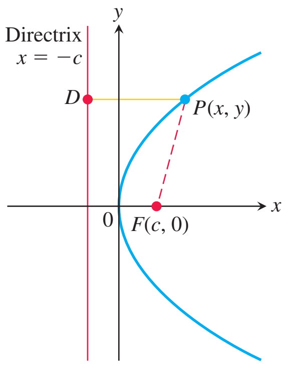



**FIGURE 11.46** The distance from the focus $F$ to any point $P$ on a parabola equals the distance from $P$ to the nearest point $D$ on the directrix, so $PF = PD$.

Whereas a parabola has one focus and one directrix, each **ellipse** has two foci and two **directrices**. These are the lines perpendicular to the major axis at distances $\pm a/e$ from the center. From Figure 11.46 we see that a parabola has the property

$$PF = 1 \cdot PD \tag{1}$$

for any point $P$ on it, where $F$ is the focus and $D$ is the point nearest $P$ on the directrix. For an ellipse, it can be shown that the equations that replace Equation (1) are

$$PF_1 = e \cdot PD_1, \qquad PF_2 = e \cdot PD_2. \tag{2}$$

Here, $e$ is the eccentricity, $P$ is any point on the ellipse, $F_1$ and $F_2$ are the foci, and $D_1$ and $D_2$ are the points on the directrices nearest $P$ (Figure 11.47).

In both Equations (2) the directrix and focus must correspond; that is, if we use the distance from $P$ to $F_1$, we must also use the distance from $P$ to the directrix at the same end of the ellipse. The directrix $x = -a/e$ corresponds to $F_1(-c, 0)$, and the directrix $x = a/e$ corresponds to $F_2(c, 0)$.

As with the ellipse, it can be shown that the lines $x = \pm a/e$ act as **directrices** for the **hyperbola** and that

$$PF_1 = e \cdot PD_1 \quad \text{and} \quad PF_2 = e \cdot PD_2. \tag{3}$$

Here $P$ is any point on the hyperbola, $F_1$ and $F_2$ are the foci, and $D_1$ and $D_2$ are the points nearest $P$ on the directrices (Figure 11.48).

In both the ellipse and the hyperbola, the eccentricity is the ratio of the distance between the foci to the distance between the vertices (because $c/a = 2c/2a$).

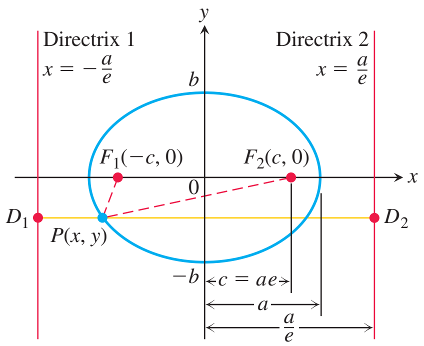



**FIGURE 11.47** The foci and directrices of the ellipse $(x^2/a^2) + (y^2/b^2) = 1$. Directrix 1 corresponds to focus $F_1$ and directrix 2 to focus $F_2$.

$$\text{Eccentricity} = \frac{\text{distance between foci}}{\text{distance between vertices}}$$

In an ellipse, the foci are closer together than the vertices and the ratio is less than 1. In a hyperbola, the foci are farther apart than the vertices and the ratio is greater than 1.

The "focus–directrix" equation $PF = e \cdot PD$ unites the parabola, ellipse, and hyperbola in the following way. Suppose that the distance $PF$ of a point $P$ from a fixed point $F$ (the focus) is a constant multiple of its distance from a fixed line (the directrix). That is, suppose

$$PF = e \cdot PD, \tag{4}$$

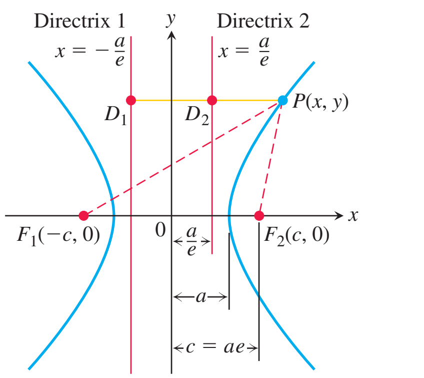



**FIGURE 11.48** The foci and directrices of the hyperbola $(x^2/a^2) - (y^2/b^2) = 1$. No matter where $P$ lies on the hyperbola, $PF_1 = e \cdot PD_1$ and $PF_2 = e \cdot PD_2$.

where $e$ is the constant of proportionality. Then the path traced by $P$ is

**(a)** a *parabola* if $e = 1$,

**(b)** an *ellipse* of eccentricity $e$ if $e < 1$, and

**(c)** a *hyperbola* of eccentricity $e$ if $e > 1$.

As $e$ increases ($e \to 1^-$), ellipses become more oblong, and ($e \to \infty$) hyperbolas flatten toward two lines parallel to the directrix. There are no coordinates in Equation (4), and when we try to translate it into Cartesian coordinate form, it translates in different ways depending on the size of $e$. However, as we are about to see, in polar coordinates the equation $PF = e \cdot PD$ translates into a single equation regardless of the value of $e$.

Given the focus and corresponding directrix of a hyperbola centered at the origin and with foci on the $x$-axis, we can use the dimensions shown in Figure 11.48 to find $e$.

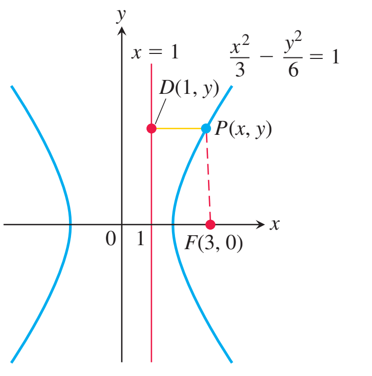



**FIGURE 11.49** The hyperbola and directrix in Example 1.

Knowing $e$, we can derive a Cartesian equation for the hyperbola from the equation $PF = e \cdot PD$, as in the next example. We can find equations for ellipses centered at the origin and with foci on the $x$-axis in a similar way, using the dimensions shown in Figure 11.47.

**EXAMPLE 1** Find a Cartesian equation for the hyperbola centered at the origin that has a focus at $(3, 0)$ and the line $x = 1$ as the corresponding directrix.

**Solution** We first use the dimensions shown in Figure 11.48 to find the hyperbola's eccentricity. The focus is (see Figure 11.49)

$$(c, 0) = (3, 0), \qquad \text{so} \qquad c = 3.$$

Again from Figure 11.48, the directrix is the line

$$x = \frac{a}{e} = 1, \qquad \text{so} \qquad a = e.$$

When combined with the equation $e = c/a$ that defines eccentricity, these results give

$$e = \frac{c}{a} = \frac{3}{e}, \qquad \text{so} \qquad e^2 = 3 \quad \text{and} \quad e = \sqrt{3}.$$

Knowing $e$, we can now derive the equation we want from the equation $PF = e \cdot PD$. In the coordinates of Figure 11.49, we have

$$PF = e \cdot PD \qquad \text{Eq. (4)}$$

$$\sqrt{(x-3)^2 + (y-0)^2} = \sqrt{3}\,|x - 1| \qquad e = \sqrt{3}$$

$$x^2 - 6x + 9 + y^2 = 3(x^2 - 2x + 1) \qquad \text{Square both sides.}$$

$$2x^2 - y^2 = 6$$

$$\frac{x^2}{3} - \frac{y^2}{6} = 1.$$

## Polar Equations

To find a polar equation for an ellipse, parabola, or hyperbola, we place one focus at the origin and the corresponding directrix to the right of the origin along the vertical line $x = k$ (Figure 11.50). In polar coordinates, this makes

$$PF = r$$

and

$$PD = k - FB = k - r\cos\theta.$$

The conic's focus–directrix equation $PF = e \cdot PD$ then becomes

$$r = e(k - r\cos\theta),$$

which can be solved for $r$ to obtain the following expression.

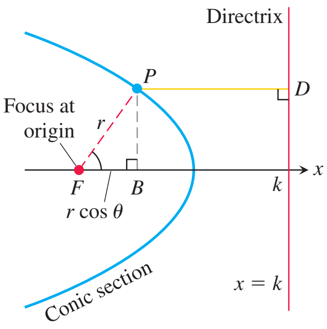



**FIGURE 11.50** If a conic section is put in the position with its focus placed at the origin and a directrix perpendicular to the initial ray and right of the origin, we can find its polar equation from the conic's focus–directrix equation.

**Polar Equation for a Conic with Eccentricity $e$**

$$r = \frac{ke}{1 + e\cos\theta}, \qquad (5)$$

where $x = k > 0$ is the vertical directrix.

**EXAMPLE 2** Here are polar equations for three conics. The eccentricity values identifying the conic are the same for both polar and Cartesian coordinates.

$$e = \frac{1}{2}: \quad \text{ellipse} \quad r = \frac{k}{2 + \cos\theta}$$

$$e = 1: \quad \text{parabola} \quad r = \frac{k}{1 + \cos\theta}$$

$$e = 2: \quad \text{hyperbola} \quad r = \frac{2k}{1 + 2\cos\theta}$$

You may see variations of Equation (5), depending on the location of the directrix. If the directrix is the line $x = -k$ to the left of the origin (the origin is still a focus), we replace Equation (5) with

$$r = \frac{ke}{1 - e\cos\theta}.$$

The denominator now has a $(-)$ instead of a $(+)$. If the directrix is either of the lines $y = k$ or $y = -k$, the equations have sines in them instead of cosines, as shown in Figure 11.51.

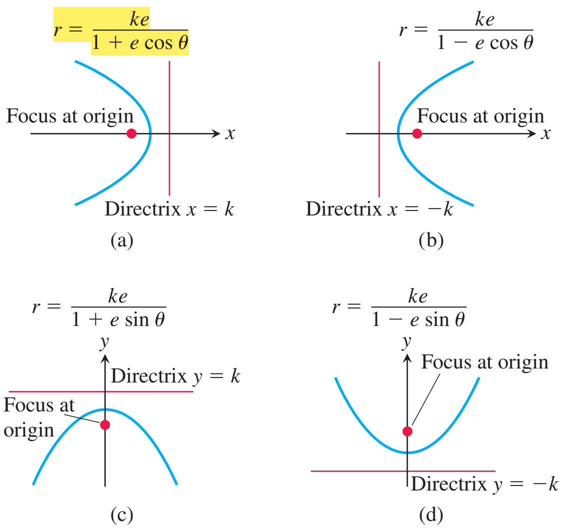



**FIGURE 11.51** Equations for conic sections with eccentricity $e > 0$ but different locations of the directrix. The graphs here show a parabola, so $e = 1$.

**EXAMPLE 3** Find an equation for the hyperbola with eccentricity $3/2$ and directrix $x = 2$.

**Solution** We use Equation (5) with $k = 2$ and $e = 3/2$:

$$r = \frac{2(3/2)}{1 + (3/2)\cos\theta} \quad \text{or} \quad r = \frac{6}{2 + 3\cos\theta}.$$

**EXAMPLE 4** Find the directrix of the parabola

$$r = \frac{25}{10 + 10\cos\theta}.$$

**Solution** We divide the numerator and denominator by 10 to put the equation in standard polar form:

$$r = \frac{5/2}{1 + \cos\theta}.$$

This is the equation

$$r = \frac{ke}{1 + e\cos\theta}$$

with $k = 5/2$ and $e = 1$. The equation of the directrix is $x = 5/2$. ■

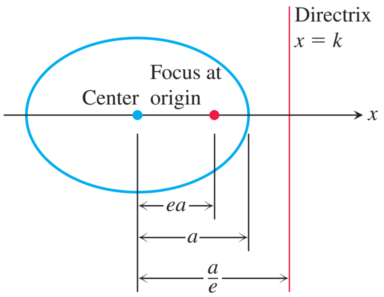



**FIGURE 11.52** In an ellipse with semimajor axis $a$, the focus–directrix distance is $k = (a/e) - ea$, so $ke = a(1 - e^2)$.

From the ellipse diagram in Figure 11.52, we see that $k$ is related to the eccentricity $e$ and the semimajor axis $a$ by the equation

$$k = \frac{a}{e} - ea.$$

From this, we find that $ke = a(1 - e^2)$. Replacing $ke$ in Equation (5) by $a(1 - e^2)$ gives the standard polar equation for an ellipse.

> **Polar Equation for the Ellipse with Eccentricity $e$ and Semimajor Axis $a$**
>
> $$r = \frac{a(1 - e^2)}{1 + e\cos\theta} \qquad (6)$$

Notice that when $e = 0$, Equation (6) becomes $r = a$, which represents a circle.

## Lines

Suppose the perpendicular from the origin to line $L$ meets $L$ at the point $P_0(r_0, \theta_0)$, with $r_0 \geq 0$ (Figure 11.53). Then, if $P(r, \theta)$ is any other point on $L$, the points $P$, $P_0$, and $O$ are the vertices of a right triangle, from which we can read the relation

$$r_0 = r\cos(\theta - \theta_0).$$

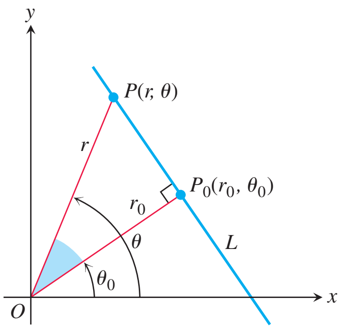



**FIGURE 11.53** We can obtain a polar equation for line $L$ by reading the relation $r_0 = r\cos(\theta - \theta_0)$ from the right triangle $OP_0P$.

> **The Standard Polar Equation for Lines**
>
> If the point $P_0(r_0, \theta_0)$ is the foot of the perpendicular from the origin to the line $L$, and $r_0 \geq 0$, then an equation for $L$ is
>
> $$r\cos(\theta - \theta_0) = r_0. \qquad (7)$$

For example, if $\theta_0 = \pi/3$ and $r_0 = 2$, we find that

$$r\cos\left(\theta - \frac{\pi}{3}\right) = 2$$

$$r\left(\cos\theta\cos\frac{\pi}{3} + \sin\theta\sin\frac{\pi}{3}\right) = 2$$

$$\frac{1}{2}r\cos\theta + \frac{\sqrt{3}}{2}r\sin\theta = 2, \qquad \text{or} \qquad x + \sqrt{3}\,y = 4.$$

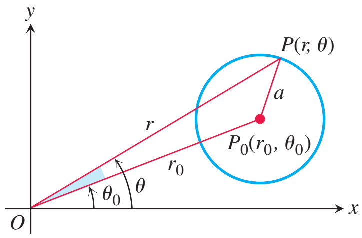



**FIGURE 11.54** We can get a polar equation for this circle by applying the Law of Cosines to triangle $OP_0P$.

## Circles

To find a polar equation for the circle of radius $a$ centered at $P_0(r_0, \theta_0)$, we let $P(r, \theta)$ be a point on the circle and apply the Law of Cosines to triangle $OP_0P$ (Figure 11.54). This gives

$$a^2 = r_0{}^2 + r^2 - 2r_0 r \cos(\theta - \theta_0).$$

If the circle passes through the origin, then $r_0 = a$ and this equation simplifies to

$$\begin{aligned} a^2 &= a^2 + r^2 - 2ar\cos(\theta - \theta_0) \\ r^2 &= 2ar\cos(\theta - \theta_0) \\ r &= 2a\cos(\theta - \theta_0). \end{aligned}$$

If the circle's center lies on the positive $x$-axis, $\theta_0 = 0$ and we get the further simplification

$$r = 2a\cos\theta. \tag{8}$$

If the center lies on the positive $y$-axis, $\theta = \pi/2$, $\cos(\theta - \pi/2) = \sin\theta$, and the equation $r = 2a\cos(\theta - \theta_0)$ becomes

$$r = 2a\sin\theta. \tag{9}$$

Equations for circles through the origin centered on the negative $x$- and $y$-axes can be obtained by replacing $r$ with $-r$ in the above equations.

**EXAMPLE 5** Here are several polar equations given by Equations (8) and (9) for circles through the origin and having centers that lie on the $x$- or $y$-axis.

| Radius | Center (polar coordinates) | Polar equation |
|---|---|---|
| 3 | $(3, 0)$ | $r = 6\cos\theta$ |
| 2 | $(2, \pi/2)$ | $r = 4\sin\theta$ |
| 1/2 | $(-1/2, 0)$ | $r = -\cos\theta$ |
| 1 | $(-1, \pi/2)$ | $r = -2\sin\theta$ |

## Exercises 11.7

### Ellipses and Eccentricity

In Exercises 1–8, find the eccentricity of the ellipse. Then find and graph the ellipse's foci and directrices.

1. $16x^2 + 25y^2 = 400$
2. $7x^2 + 16y^2 = 112$
3. $2x^2 + y^2 = 2$
4. $2x^2 + y^2 = 4$
5. $3x^2 + 2y^2 = 6$
6. $9x^2 + 10y^2 = 90$
7. $6x^2 + 9y^2 = 54$
8. $169x^2 + 25y^2 = 4225$

Exercises 9–12 give the foci or vertices and the eccentricities of ellipses centered at the origin of the $xy$-plane. In each case, find the ellipse's standard-form equation in Cartesian coordinates.

9. Foci: $(0, \pm 3)$ Eccentricity: 0.5
10. Foci: $(\pm 8, 0)$ Eccentricity: 0.2
11. Vertices: $(0, \pm 70)$ Eccentricity: 0.1
12. Vertices: $(\pm 10, 0)$ Eccentricity: 0.24

Exercises 13–16 give foci and corresponding directrices of ellipses centered at the origin of the $xy$-plane. In each case, use the dimensions in Figure 11.47 to find the eccentricity of the ellipse. Then find the ellipse's standard-form equation in Cartesian coordinates.

13. Focus: $(\sqrt{5}, 0)$ Directrix: $x = \dfrac{9}{\sqrt{5}}$
14. Focus: $(4, 0)$ Directrix: $x = \dfrac{16}{3}$
15. Focus: $(-4, 0)$ Directrix: $x = -16$
16. Focus: $(-\sqrt{2}, 0)$ Directrix: $x = -2\sqrt{2}$

## Hyperbolas and Eccentricity

In Exercises 17–24, find the eccentricity of the hyperbola. Then find and graph the hyperbola's foci and directrices.

**17.** $x^2 - y^2 = 1$

**18.** $9x^2 - 16y^2 = 144$

**19.** $y^2 - x^2 = 8$

**20.** $y^2 - x^2 = 4$

**21.** $8x^2 - 2y^2 = 16$

**22.** $y^2 - 3x^2 = 3$

**23.** $8y^2 - 2x^2 = 16$

**24.** $64x^2 - 36y^2 = 2304$

Exercises 25–28 give the eccentricities and the vertices or foci of hyperbolas centered at the origin of the $xy$-plane. In each case, find the hyperbola's standard-form equation in Cartesian coordinates.

**25.** Eccentricity: 3
Vertices: $(0, \pm 1)$

**26.** Eccentricity: 2
Vertices: $(\pm 2, 0)$

**27.** Eccentricity: 3
Foci: $(\pm 3, 0)$

**28.** Eccentricity: 1.25
Foci: $(0, \pm 5)$

## Eccentricities and Directrices

Exercises 29–36 give the eccentricities of conic sections with one focus at the origin along with the directrix corresponding to that focus. Find a polar equation for each conic section.

**29.** $e = 1, \quad x = 2$

**30.** $e = 1, \quad y = 2$

**31.** $e = 5, \quad y = -6$

**32.** $e = 2, \quad x = 4$

**33.** $e = 1/2, \quad x = 1$

**34.** $e = 1/4, \quad x = -2$

**35.** $e = 1/5, \quad y = -10$

**36.** $e = 1/3, \quad y = 6$

## Parabolas and Ellipses

Sketch the parabolas and ellipses in Exercises 37–44. Include the directrix that corresponds to the focus at the origin. Label the vertices with appropriate polar coordinates. Label the centers of the ellipses as well.

**37.** $r = \dfrac{1}{1 + \cos\theta}$

**38.** $r = \dfrac{6}{2 + \cos\theta}$

**39.** $r = \dfrac{25}{10 - 5\cos\theta}$

**40.** $r = \dfrac{4}{2 - 2\cos\theta}$

**41.** $r = \dfrac{400}{16 + 8\sin\theta}$

**42.** $r = \dfrac{12}{3 + 3\sin\theta}$

**43.** $r = \dfrac{8}{2 - 2\sin\theta}$

**44.** $r = \dfrac{4}{2 - \sin\theta}$

## Lines

Sketch the lines in Exercises 45–48 and find Cartesian equations for them.

**45.** $r\cos\left(\theta - \dfrac{\pi}{4}\right) = \sqrt{2}$

**46.** $r\cos\left(\theta + \dfrac{3\pi}{4}\right) = 1$

**47.** $r\cos\left(\theta - \dfrac{2\pi}{3}\right) = 3$

**48.** $r\cos\left(\theta + \dfrac{\pi}{3}\right) = 2$

Find a polar equation in the form $r\cos(\theta - \theta_0) = r_0$ for each of the lines in Exercises 49–52.

**49.** $\sqrt{2}x + \sqrt{2}y = 6$

**50.** $\sqrt{3}x - y = 1$

**51.** $y = -5$

**52.** $x = -4$

## Circles

Sketch the circles in Exercises 53–56. Give polar coordinates for their centers and identify their radii.

**53.** $r = 4\cos\theta$

**54.** $r = 6\sin\theta$

**55.** $r = -2\cos\theta$

**56.** $r = -8\sin\theta$

Find polar equations for the circles in Exercises 57–64. Sketch each circle in the coordinate plane and label it with both its Cartesian and polar equations.

**57.** $(x - 6)^2 + y^2 = 36$

**58.** $(x + 2)^2 + y^2 = 4$

**59.** $x^2 + (y - 5)^2 = 25$

**60.** $x^2 + (y + 7)^2 = 49$

**61.** $x^2 + 2x + y^2 = 0$

**62.** $x^2 - 16x + y^2 = 0$

**63.** $x^2 + y^2 + y = 0$

**64.** $x^2 + y^2 - \frac{4}{3}y = 0$

## Examples of Polar Equations

T Graph the lines and conic sections in Exercises 65–74.

**65.** $r = 3\sec(\theta - \pi/3)$

**66.** $r = 4\sec(\theta + \pi/6)$

**67.** $r = 4\sin\theta$

**68.** $r = -2\cos\theta$

**69.** $r = 8/(4 + \cos\theta)$

**70.** $r = 8/(4 + \sin\theta)$

**71.** $r = 1/(1 - \sin\theta)$

**72.** $r = 1/(1 + \cos\theta)$

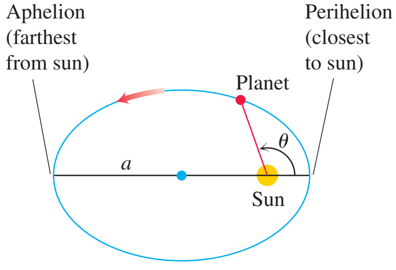
**73.** $r = 1/(1 + 2\sin\theta)$

**74.** $r = 1/(1 + 2\cos\theta)$

**75. Perihelion and aphelion** A planet travels about its sun in an ellipse whose semimajor axis has length $a$. (See accompanying figure.)

**a.** Show that $r = a(1 - e)$ when the planet is closest to the sun and that $r = a(1 + e)$ when the planet is farthest from the sun.

**b.** Use the data in the table in Exercise 76 to find how close each planet in our solar system comes to the sun and how far away each planet gets from the sun.




**76. Planetary orbits** Use the data in the table below and Equation (6) to find polar equations for the orbits of the planets.

| Planet | Semimajor axis (astronomical units) | Eccentricity |
|---|---|---|
| Mercury | 0.3871 | 0.2056 |
| Venus | 0.7233 | 0.0068 |
| Earth | 1.000 | 0.0167 |
| Mars | 1.524 | 0.0934 |
| Jupiter | 5.203 | 0.0484 |
| Saturn | 9.539 | 0.0543 |
| Uranus | 19.18 | 0.0460 |
| Neptune | 30.06 | 0.0082 |

# Chapter 11 Questions to Guide Your Review

1. What is a parametrization of a curve in the $xy$-plane? Does a function $y = f(x)$ always have a parametrization? Are parametrizations of a curve unique? Give examples.
2. Give some typical parametrizations for lines, circles, parabolas, ellipses, and hyperbolas. How might the parametrized curve differ from the graph of its Cartesian equation?
3. What is a cycloid? What are typical parametric equations for cycloids? What physical properties account for the importance of cycloids?
4. What is the formula for the slope $dy/dx$ of a parametrized curve $x = f(t)$, $y = g(t)$? When does the formula apply? When can you expect to be able to find $d^2y/dx^2$ as well? Give examples.
5. How can you sometimes find the area bounded by a parametrized curve and one of the coordinate axes?
6. How do you find the length of a smooth parametrized curve $x = f(t)$, $y = g(t)$, $a \le t \le b$? What does smoothness have to do with length? What else do you need to know about the parametrization in order to find the curve's length? Give examples.
7. What is the arc length function for a smooth parametrized curve? What is its arc length differential?
8. Under what conditions can you find the area of the surface generated by revolving a curve $x = f(t)$, $y = g(t)$, $a \le t \le b$, about the $x$-axis? the $y$-axis? Give examples.
9. What are polar coordinates? What equations relate polar coordinates to Cartesian coordinates? Why might you want to change from one coordinate system to the other?
10. What consequence does the lack of uniqueness of polar coordinates have for graphing? Give an example.
11. How do you graph equations in polar coordinates? Include in your discussion symmetry, slope, behavior at the origin, and the use of Cartesian graphs. Give examples.
12. How do you find the area of a region $0 \le r_1(\theta) \le r \le r_2(\theta)$, $\alpha \le \theta \le \beta$, in the polar coordinate plane? Give examples.
13. Under what conditions can you find the length of a curve $r = f(\theta)$, $\alpha \le \theta \le \beta$, in the polar coordinate plane? Give an example of a typical calculation.
14. What is a parabola? What are the Cartesian equations for parabolas whose vertices lie at the origin and whose foci lie on the coordinate axes? How can you find the focus and directrix of such a parabola from its equation?
15. What is an ellipse? What are the Cartesian equations for ellipses centered at the origin with foci on one of the coordinate axes? How can you find the foci, vertices, and directrices of such an ellipse from its equation?
16. What is a hyperbola? What are the Cartesian equations for hyperbolas centered at the origin with foci on one of the coordinate axes? How can you find the foci, vertices, and directrices of such an ellipse from its equation?
17. What is the eccentricity of a conic section? How can you classify conic sections by eccentricity? How does eccentricity change the shape of ellipses and hyperbolas?
18. Explain the equation $PF = e \cdot PD$.
19. What are the standard equations for lines and conic sections in polar coordinates? Give examples.

# Chapter 11 Practice Exercises

## Identifying Parametric Equations in the Plane

Exercises 1–6 give parametric equations and parameter intervals for the motion of a particle in the $xy$-plane. Identify the particle's path by finding a Cartesian equation for it. Graph the Cartesian equation and indicate the direction of motion and the portion traced by the particle.

1. $x = t/2, \quad y = t + 1; \quad -\infty < t < \infty$
2. $x = \sqrt{t}, \quad y = 1 - \sqrt{t}; \quad t \ge 0$
3. $x = (1/2)\tan t, \quad y = (1/2)\sec t; \quad -\pi/2 < t < \pi/2$
4. $x = -2\cos t, \quad y = 2\sin t; \quad 0 \le t \le \pi$
5. $x = -\cos t, \quad y = \cos^2 t; \quad 0 \le t \le \pi$
6. $x = 4\cos t, \quad y = 9\sin t; \quad 0 \le t \le 2\pi$

## Finding Parametric Equations and Tangent Lines

7. Find parametric equations and a parameter interval for the motion of a particle in the $xy$-plane that traces the ellipse $16x^2 + 9y^2 = 144$ once counterclockwise. (There are many ways to do this.)
8. Find parametric equations and a parameter interval for the motion of a particle that starts at the point $(-2, 0)$ in the $xy$-plane and traces the circle $x^2 + y^2 = 4$ three times clockwise. (There are many ways to do this.)

In Exercises 9 and 10, find an equation for the line in the $xy$-plane that is tangent to the curve at the point corresponding to the given value of $t$. Also, find the value of $d^2y/dx^2$ at this point.

9. $x = (1/2)\tan t, \quad y = (1/2)\sec t; \quad t = \pi/3$
10. $x = 1 + 1/t^2, \quad y = 1 - 3/t; \quad t = 2$
11. Eliminate the parameter to express the curve in the form $y = f(x)$.
    **a.** $x = 4t^2, \quad y = t^3 - 1$
    **b.** $x = \cos t, \quad y = \tan t$
12. Find parametric equations for the given curve.
    **a.** Line through $(1, -2)$ with slope 3
    **b.** $(x - 1)^2 + (y + 2)^2 = 9$
    **c.** $y = 4x^2 - x$
    **d.** $9x^2 + 4y^2 = 36$

## Lengths of Curves

Find the lengths of the curves in Exercises 13–19.

**13.** $y = x^{1/2} - (1/3)x^{3/2}, \quad 1 \le x \le 4$

**14.** $x = y^{2/3}, \quad 1 \le y \le 8$

**15.** $y = (5/12)x^{6/5} - (5/8)x^{4/5}, \quad 1 \le x \le 32$

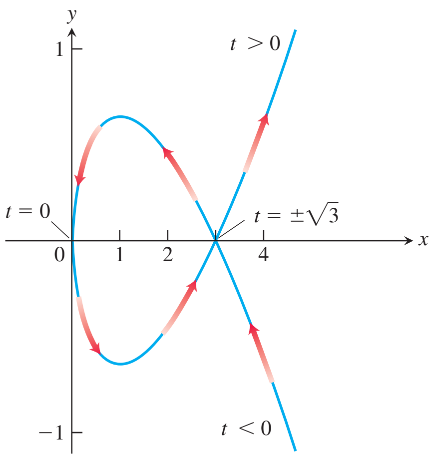
**16.** $x = \left(y^3/12\right) + (1/y), \quad 1 \le y \le 2$

**17.** $x = 5\cos t - \cos 5t, \quad y = 5\sin t - \sin 5t, \quad 0 \le t \le \pi/2$

**18.** $x = t^3 - 6t^2, \quad y = t^3 + 6t^2, \quad 0 \le t \le 1$

**19.** $x = 3\cos\theta, \quad y = 3\sin\theta, \quad 0 \le \theta \le \dfrac{3\pi}{2}$

**20.** Find the length of the enclosed loop $x = t^2, y = \left(t^3/3\right) - t$ shown here. The loop starts at $t = -\sqrt{3}$ and ends at $t = \sqrt{3}$.




## Surface Areas

Find the areas of the surfaces generated by revolving the curves in Exercises 21 and 22 about the indicated axes.

**21.** $x = t^2/2, \quad y = 2t, \quad 0 \le t \le \sqrt{5}; \quad x\text{-axis}$

**22.** $x = t^2 + 1/(2t), \quad y = 4\sqrt{t}, \quad 1/\sqrt{2} \le t \le 1; \quad y\text{-axis}$

## Polar to Cartesian Equations

Sketch the lines in Exercises 23–28. Also, find a Cartesian equation for each line.

**23.** $r\cos\left(\theta + \dfrac{\pi}{3}\right) = 2\sqrt{3}$

**24.** $r\cos\left(\theta - \dfrac{3\pi}{4}\right) = \dfrac{\sqrt{2}}{2}$

**25.** $r = 2\sec\theta$

**26.** $r = -\sqrt{2}\sec\theta$

**27.** $r = -(3/2)\csc\theta$

**28.** $r = \left(3\sqrt{3}\right)\csc\theta$

Find Cartesian equations for the circles in Exercises 29–32. Sketch each circle in the coordinate plane and label it with both its Cartesian and polar equations.

**29.** $r = -4\sin\theta$

**30.** $r = 3\sqrt{3}\sin\theta$

**31.** $r = 2\sqrt{2}\cos\theta$

**32.** $r = -6\cos\theta$

## Cartesian to Polar Equations

Find polar equations for the circles in Exercises 33–36. Sketch each circle in the coordinate plane and label it with both its Cartesian and polar equations.

**33.** $x^2 + y^2 + 5y = 0$

**34.** $x^2 + y^2 - 2y = 0$

**35.** $x^2 + y^2 - 3x = 0$

**36.** $x^2 + y^2 + 4x = 0$

## Graphs in Polar Coordinates

Sketch the regions defined by the polar coordinate inequalities in Exercises 37 and 38.

**37.** $0 \le r \le 6\cos\theta$

**38.** $-4\sin\theta \le r \le 0$

Match each graph in Exercises 39–46 with the appropriate equation (a)–(l). There are more equations than graphs, so some equations will not be matched.

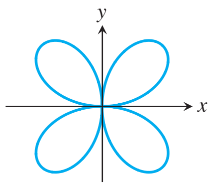
**a.** $r = \cos 2\theta$    **b.** $r\cos\theta = 1$    **c.** $r = \dfrac{6}{1 - 2\cos\theta}$

**d.** $r = \sin 2\theta$    **e.** $r = \theta$    **f.** $r^2 = \cos 2\theta$

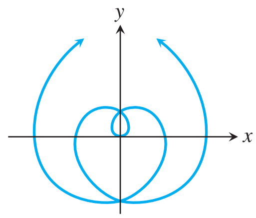
**g.** $r = 1 + \cos\theta$    **h.** $r = 1 - \sin\theta$    **i.** $r = \dfrac{2}{1 - \cos\theta}$

**j.** $r^2 = \sin 2\theta$    **k.** $r = -\sin\theta$    **l.** $r = 2\cos\theta + 1$

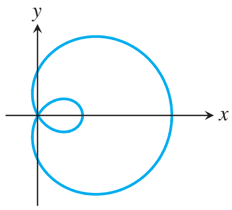
**39.** Four-leaved rose



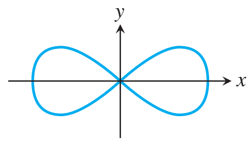

**40.** Spiral



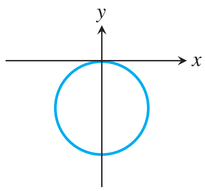

**41.** Limaçon



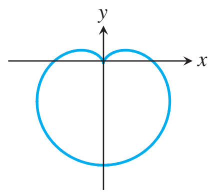

**42.** Lemniscate



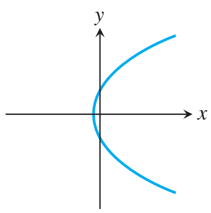

**43.** Circle



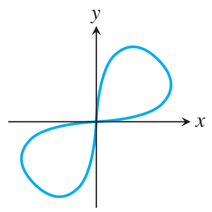

**44.** Cardioid




**45.** Parabola




**46.** Lemniscate




## Area in Polar Coordinates

Find the areas of the regions in the polar coordinate plane described in Exercises 47–50.

**47.** Enclosed by the limaçon $r = 2 - \cos\theta$

**48.** Enclosed by one leaf of the three-leaved rose $r = \sin 3\theta$

**49.** Inside the "figure eight" $r = 1 + \cos 2\theta$ and outside the circle $r = 1$

**50.** Inside the cardioid $r = 2(1 + \sin\theta)$ and outside the circle $r = 2\sin\theta$

### Length in Polar Coordinates

Find the lengths of the curves given by the polar coordinate equations in Exercises 51–54.

**51.** $r = -1 + \cos\theta$

**52.** $r = 2\sin\theta + 2\cos\theta, \quad 0 \le \theta \le \pi/2$

**53.** $r = 8\sin^3(\theta/3), \quad 0 \le \theta \le \pi/4$

**54.** $r = \sqrt{1 + \cos 2\theta}, \quad -\pi/2 \le \theta \le \pi/2$

### Graphing Conic Sections

Sketch the parabolas in Exercises 55–58. Include the focus and directrix in each sketch.

**55.** $x^2 = -4y$

**56.** $x^2 = 2y$

**57.** $y^2 = 3x$

**58.** $y^2 = -(8/3)x$

Find the eccentricities of the ellipses and hyperbolas in Exercises 59–62. Sketch each conic section. Include the foci, vertices, and asymptotes (as appropriate) in your sketch.

**59.** $16x^2 + 7y^2 = 112$

**60.** $x^2 + 2y^2 = 4$

**61.** $3x^2 - y^2 = 3$

**62.** $5y^2 - 4x^2 = 20$

Exercises 63–68 give equations for conic sections and tell how many units up or down and to the right or left each curve is to be shifted. Find an equation for the new conic section, and find the new foci, vertices, centers, and asymptotes, as appropriate. If the curve is a parabola, find the new directrix as well.

**63.** $x^2 = -12y$, right 2, up 3

**64.** $y^2 = 10x$, left 1/2, down 1

**65.** $\dfrac{x^2}{9} + \dfrac{y^2}{25} = 1$, left 3, down 5

**66.** $\dfrac{x^2}{169} + \dfrac{y^2}{144} = 1$, right 5, up 12

**67.** $\dfrac{y^2}{8} - \dfrac{x^2}{2} = 1$, right 2, up $2\sqrt{2}$

**68.** $\dfrac{x^2}{36} - \dfrac{y^2}{64} = 1$, left 10, down 3

### Identifying Conic Sections

Complete the squares to identify the conic sections in Exercises 69–76. Find their foci, vertices, centers, and asymptotes (as appropriate). If the curve is a parabola, find its directrix as well.

**69.** $x^2 - 4x - 4y^2 = 0$

**70.** $4x^2 - y^2 + 4y = 8$

**71.** $y^2 - 2y + 16x = -49$

**72.** $x^2 - 2x + 8y = -17$

**73.** $9x^2 + 16y^2 + 54x - 64y = -1$

**74.** $25x^2 + 9y^2 - 100x + 54y = 44$

**75.** $x^2 + y^2 - 2x - 2y = 0$

**76.** $x^2 + y^2 + 4x + 2y = 1$

### Conics in Polar Coordinates

Sketch the conic sections whose polar coordinate equations are given in Exercises 77–80. Give polar coordinates for the vertices and, in the case of ellipses, for the centers as well.

**77.** $r = \dfrac{2}{1 + \cos\theta}$

**78.** $r = \dfrac{8}{2 + \cos\theta}$

**79.** $r = \dfrac{6}{1 - 2\cos\theta}$

**80.** $r = \dfrac{12}{3 + \sin\theta}$

Exercises 81–84 give the eccentricities of conic sections with one focus at the origin of the polar coordinate plane, along with the directrix for that focus. Find a polar equation for each conic section.

**81.** $e = 2, \quad r\cos\theta = 2$

**82.** $e = 1, \quad r\cos\theta = -4$

**83.** $e = 1/2, \quad r\sin\theta = 2$

**84.** $e = 1/3, \quad r\sin\theta = -6$

### Theory and Examples

**85.** Find the volume of the solid generated by revolving the region enclosed by the ellipse $9x^2 + 4y^2 = 36$ about **(a)** the $x$-axis, **(b)** the $y$-axis.

**86.** The "triangular" region in the first quadrant bounded by the $x$-axis, the line $x = 4$, and the hyperbola $9x^2 - 4y^2 = 36$ is revolved about the $x$-axis to generate a solid. Find the volume of the solid.

**87.** Show that the equations $x = r\cos\theta$, $y = r\sin\theta$ transform the polar equation

$$r = \frac{k}{1 + e\cos\theta}$$

into the Cartesian equation

$$(1 - e^2)x^2 + y^2 + 2kex - k^2 = 0.$$

**88. Archimedes spirals** The graph of an equation of the form $r = a\theta$, where $a$ is a nonzero constant, is called an *Archimedes spiral*. Is there anything special about the widths between the successive turns of such a spiral?

## Chapter 11 Additional and Advanced Exercises

### Finding Conic Sections

**1.** Find an equation for the parabola with focus (4, 0) and directrix $x = 3$. Sketch the parabola together with its vertex, focus, and directrix.

**2.** Find the vertex, focus, and directrix of the parabola

$$x^2 - 6x - 12y + 9 = 0.$$

**3.** Find an equation for the curve traced by the point $P(x, y)$ if the distance from $P$ to the vertex of the parabola $x^2 = 4y$ is twice the distance from $P$ to the focus. Identify the curve.

**4.** A line segment of length $a + b$ runs from the $x$-axis to the $y$-axis. The point $P$ on the segment lies $a$ units from one end and $b$ units from the other end. Show that $P$ traces an ellipse as the ends of the segment slide along the axes.

**5.** The vertices of an ellipse of eccentricity 0.5 lie at the points $(0, \pm 2)$. Where do the foci lie?

**6.** Find an equation for the ellipse of eccentricity $2/3$ that has the line $x = 2$ as a directrix and the point (4, 0) as the corresponding focus.

**7.** One focus of a hyperbola lies at the point $(0, -7)$ and the corresponding directrix is the line $y = -1$. Find an equation for the hyperbola if its eccentricity is **(a)** 2, **(b)** 5.

**8.** Find an equation for the hyperbola with foci $(0, -2)$ and $(0, 2)$ that passes through the point $(12, 7)$.

**9.** Show that the line

$$b^2xx_1 + a^2yy_1 - a^2b^2 = 0$$

is tangent to the ellipse $b^2x^2 + a^2y^2 - a^2b^2 = 0$ at the point $(x_1, y_1)$ on the ellipse.

**10.** Show that the line

$$b^2xx_1 - a^2yy_1 - a^2b^2 = 0$$

is tangent to the hyperbola $b^2x^2 - a^2y^2 - a^2b^2 = 0$ at the point $(x_1, y_1)$ on the hyperbola.

### Equations and Inequalities

What points in the $xy$-plane satisfy the equations and inequalities in Exercises 11–16? Draw a figure for each exercise.

**11.** $(x^2 - y^2 - 1)(x^2 + y^2 - 25)(x^2 + 4y^2 - 4) = 0$

**12.** $(x + y)(x^2 + y^2 - 1) = 0$

**13.** $(x^2/9) + (y^2/16) \le 1$

**14.** $(x^2/9) - (y^2/16) \le 1$

**15.** $(9x^2 + 4y^2 - 36)(4x^2 + 9y^2 - 16) \le 0$

**16.** $(9x^2 + 4y^2 - 36)(4x^2 + 9y^2 - 16) > 0$

### Polar Coordinates

**17. a.** Find an equation in polar coordinates for the curve

$$x = e^{2t}\cos t, \quad y = e^{2t}\sin t; \quad -\infty < t < \infty.$$

**b.** Find the length of the curve from $t = 0$ to $t = 2\pi$.

**18.** Find the length of the curve $r = 2\sin^3(\theta/3)$, $0 \le \theta \le 3\pi$, in the polar coordinate plane.

Exercises 19–22 give the eccentricities of conic sections with one focus at the origin of the polar coordinate plane, along with the directrix for that focus. Find a polar equation for each conic section.

**19.** $e = 2, \quad r\cos\theta = 2$

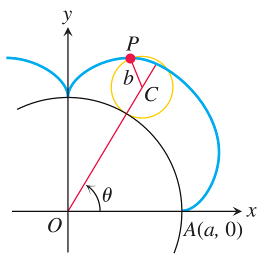
**20.** $e = 1, \quad r\cos\theta = -4$

**21.** $e = 1/2, \quad r\sin\theta = 2$

**22.** $e = 1/3, \quad r\sin\theta = -6$

### Theory and Examples

**23. Epicycloids** When a circle rolls externally along the circumference of a second, fixed circle, any point $P$ on the circumference of the rolling circle describes an *epicycloid*, as shown here. Let the fixed circle have its center at the origin $O$ and have radius $a$.




Let the radius of the rolling circle be $b$ and let the initial position of the tracing point $P$ be $A(a, 0)$. Find parametric equations for the epicycloid, using as the parameter the angle $\theta$ from the positive $x$-axis to the line through the circles' centers.

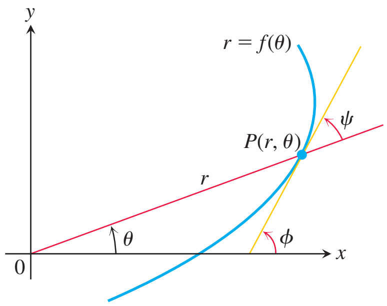
**24.** Find the centroid of the region enclosed by the $x$-axis and the cycloid arch

$$x = a(t - \sin t), \quad y = a(1 - \cos t); \quad 0 \le t \le 2\pi.$$

**The Angle Between the Radius Vector and the Tangent Line to a Polar Coordinate Curve** In Cartesian coordinates, when we want to discuss the direction of a curve at a point, we use the angle $\phi$ measured counterclockwise from the positive $x$-axis to the tangent line. In polar coordinates, it is more convenient to calculate the angle $\psi$ from the *radius vector* to the tangent line (see the accompanying figure). The angle $\phi$ can then be calculated from the relation

$$\phi = \theta + \psi, \tag{1}$$

which comes from applying the Exterior Angle Theorem to the triangle in the accompanying figure.




Suppose the equation of the curve is given in the form $r = f(\theta)$, where $f(\theta)$ is a differentiable function of $\theta$. Then

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta \tag{2}$$

are differentiable functions of $\theta$ with

$$\frac{dx}{d\theta} = -r\sin\theta + \cos\theta\frac{dr}{d\theta},$$

$$\frac{dy}{d\theta} = r\cos\theta + \sin\theta\frac{dr}{d\theta}. \tag{3}$$

Since $\psi = \phi - \theta$ from (1),

$$\tan\psi = \tan(\phi - \theta) = \frac{\tan\phi - \tan\theta}{1 + \tan\phi\tan\theta}.$$

Furthermore,

$$\tan\phi = \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$$

because $\tan\phi$ is the slope of the curve at $P$. Also,

$$\tan\theta = \frac{y}{x}.$$

Hence

$$\tan\psi = \frac{\dfrac{dy/d\theta}{dx/d\theta} - \dfrac{y}{x}}{1 + \dfrac{y}{x}\dfrac{dy/d\theta}{dx/d\theta}} = \frac{x\dfrac{dy}{d\theta} - y\dfrac{dx}{d\theta}}{x\dfrac{dx}{d\theta} + y\dfrac{dy}{d\theta}}. \tag{4}$$

The numerator in the last expression in Equation (4) is found from Equations (2) and (3) to be

$$x\frac{dy}{d\theta} - y\frac{dx}{d\theta} = r^2.$$

Similarly, the denominator is

$$x\frac{dx}{d\theta} + y\frac{dy}{d\theta} = r\frac{dr}{d\theta}.$$

When we substitute these into Equation (4), we obtain

$$\tan\psi = \frac{r}{dr/d\theta}. \tag{5}$$

This is the equation we use for finding $\psi$ as a function of $\theta$.

**25.** Show, by reference to a figure, that the angle $\beta$ between the tangents to two curves at a point of intersection may be found from the formula

$$\tan\beta = \frac{\tan\psi_2 - \tan\psi_1}{1 + \tan\psi_2 \tan\psi_1}. \tag{6}$$

When will the two curves intersect at right angles?

**26.** Find the value of $\tan\psi$ for the curve $r = \sin^4(\theta/4)$.

**27.** Find the angle between the radius vector to the curve $r = 2a\sin 3\theta$ and its tangent when $\theta = \pi/6$.

T **28. a.** Graph the hyperbolic spiral $r\theta = 1$. What appears to happen to $\psi$ as the spiral winds in around the origin?

**b.** Confirm your finding in part (a) analytically.

**29.** The circles $r = \sqrt{3}\cos\theta$ and $r = \sin\theta$ intersect at the point $(\sqrt{3}/2, \pi/3)$. Show that their tangents are perpendicular there.

**30.** Find the angle at which the cardioid $r = a(1 - \cos\theta)$ crosses the ray $\theta = \pi/2$.

# Chapter 11 Technology Application Projects

## Mathematica / Maple Modules:

***Radar Tracking of a Moving Object***

**Part I:** Convert from polar to Cartesian coordinates.

***Parametric and Polar Equations with a Figure Skater***

**Part I:** Visualize position, velocity, and acceleration to analyze motion defined by parametric equations.

**Part II:** Find and analyze the equations of motion for a figure skater tracing a polar plot.